On r-colorability of random hypergraphs

Andrey Kupavskii*, Dmitry A. Shabanov†

Abstract

The work deals with the threshold for r-colorability in the binomial model $H(n,k,p)$ of a random hypergraph. We prove that if, for some constant $\delta \in (0,1)$,

$$k^{\varphi(k)} \ln n \ll r^{k-1} \leq n^{(1-\delta)/2} \text{ and } p \leq r^{k-1} k^{-1-\varphi(k)} \frac{n}{\binom{n}{k}},$$

where $\varphi(k)$ is some function satisfying the relation $\varphi(k) = \Theta\left(\sqrt{\frac{\ln\ln k}{\ln k}}\right)$, then

$$\mathsf{P}\left(H(n,k,p) \text{ is } r\text{-colorable}\right) \to 1 \text{ as } n \to \infty.$$

This result improves the previously known results in the wide range of the parameters $r = r(n)$, $k = k(n)$.

Keywords: *random hypergraph*, *colorings of hypergraphs*, *sparse hypergraphs*, *random recoloring method*.

1 Introduction and history of the problem

The work deals with a problem concerning threshold for r-colorability in the binomial model of a random hypergraph. First of all, we recall the main definitions from the hypergraph theory.

1.1 Main definitions

A hypergraph H is a pair $H = (V, E)$, where $V = V(H)$ is some finite set (called *the vertex set* of the hypergraph) and $= E(H)$ is a collection of subsets of V, which are called *the edges* of the hypergraph. If $E \subseteq \binom{V}{k}$, i.e. every edge contains exactly k vertices, then H is called *k-uniform*. By $K_n^{(k)}$ we denote a complete k-uniform hypergraph on n vertices.

A vertex coloring of the vertex set V is called *proper* for hypergraph $H = (V, E)$ if in this coloring there is no monochromatic edges in E. A hypergraph H is called r-colorable, if there

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exists a proper coloring with r colors (r-*coloring*) for H. *The chromatic number* $\chi(H)$ of a hypergraph H is the minimum r such that H is r-colorable.

Let v be a vertex of a hypergraph H. *The degree* of v in H is the number of edges of H containing v. By $\Delta(H)$ we denote the maximum vertex degree of the hypergraph H. A hypergraph $H = (V, E)$ is called l-*simple*, if every two of its distinct edges do not share more than l common vertices, i.e.

$$\forall\, e, f \in E,\ f \neq e:\ |e \cap f| \leq l.$$

A 1-simple hypergraph is usually called *simple* hypergraph. A *cycle of length* 3 (3-*cycle* or *triangle*) in the hypergraph H is a unordered set of three distinct edges (e, f, h) such that $(e \cap f) \setminus h \neq \emptyset$, $(e \cap h) \setminus f \neq \emptyset$, $(h \cap f) \setminus e \neq \emptyset$.

In this article we study the binomial model of a random hypergraph. Given integers $n > k \geq 2$ and a real number $p \in [0, 1]$, *random hypergraph* $H(n, k, p)$ is a random spanning subhypergraph of the complete k-uniform hypergraph $K_n^{(k)}$ with the following distribution: for any spanning subhypergraph F of $K_n^{(k)}$,

$$\mathsf{P}\left(H(n, k, p) = F\right) = p^{|E(F)|}(1 - p)^{\binom{n}{k} - |E(F)|}.$$

This definition immediately implies that every edge of $K_n^{(k)}$ is included in $H(n, k, p)$ independently with equal probability p.

Suppose $\mathcal{Q}_n$ is a property of k-uniform hypergraphs on n vertices. We say that $\mathcal{Q}_n$ is an *increasing property* if, for any two k-uniform hypergraphs H and G on n vertices, H has property $\mathcal{Q}_n$ and $E(H) \subseteq E(G)$ imply that G has property $\mathcal{Q}_n$. For given function $k = k(n) \geq 2$, the function $p^* = p^*(n)$ is said to be a *threshold* (or a *threshold probability*) for an increasing property $\mathcal{Q}_n$, if

- for any $p = p(n)$ such that $p \ll p^*$, $\mathsf{P}\left(H(n, k, p) \text{ has property } \mathcal{Q}_n\right) \to 0$ as $n \to \infty$;
- for any $p = p(n)$ such that $p \gg p^*$, $\mathsf{P}\left(H(n, k, p) \text{ has property } \mathcal{Q}_n\right) \to 1$ as $n \to \infty$.

It follows from general results of B. Bollobás and A. Thomason (see [1]) concerning monotone properties of random subsets that for any function $k = k(n) \geq 2$ and any increasing property $\mathcal{Q}_n$, the threshold probability exists. In this article we are concentrated on the estimating the threshold probability for r-colorability of $H(n, k, p)$, i.e. for an increasing property $\mathcal{Q}_n = \{\text{hypergraph is not } r\text{-colorable}\}$, where $r = r(n) \geq 2$ is some function of n. In the next paragraph we shall give a background of this problem.

1.2 Previous results

The r-colorability of random hypergraph $H(n, k, p)$ was most intensively studied in the case of fixed k and $r = 2$. It appears that in this case the transition from 2-colorability to non-2-colorability is *sharp*. It follows form the general results of E. Freidgut (see [2]) that for any $k \geq 3$, there exists a sequence $d_k(n)$ such that for any $\varepsilon > 0$,

- if $p \leq (d_k(n) - \varepsilon)n/\binom{n}{k}$, then $\mathsf{P}\left(H(n, k, p) \text{ is 2-colorable}\right) \to 1$,

- but if $p \geq (d_k(n)+\varepsilon)n/\binom{n}{k}$, then $\mathsf{P}\left(H(n,k,p) \text{ is 2-colorable}\right) \to 0$.

It is widely believed that $d_k(n)$ can be replaced by a constant d_k.

First bounds for the threshold probability of 2-colorability were obtained by N. Alon and J. Spencer. They showed (see [3]) that there is a positive absolute constant c such that

$$\text{if } k \geq k_0 \text{ is fixed } \text{ and } p \leq c\,\frac{2^{k-1}}{k^2}\,\frac{n}{\binom{n}{k}},\ \text{then } \mathsf{P}\left(H(n,k,p) \text{ is 2-colorable}\right) \to 1, \tag{1}$$

$$\text{if } k \geq 3,\ \varepsilon > 0 \text{ are fixed } \text{ and } p \geq (1+\varepsilon)\,2^{k-1}\ln 2\,\frac{n}{\binom{n}{k}},$$
$$\text{then } \mathsf{P}\left(H(n,k,p) \text{ is 2-colorable}\right) \to 0. \tag{2}$$

The gap between upper and lower bounds in (1) and (2) was reduced by D. Achlioptas, J.H. Kim, M. Krivelevich and P. Tetali from the order k^2 to the order k. They proved (see [4]) that for any fixed $k \geq 3$,

$$\text{if } p \leq \frac{1}{25}\,\frac{2^{k-1}}{k}\,\frac{n}{\binom{n}{k}},\ \text{then } \mathsf{P}\left(H(n,k,p) \text{ is 2-colorable}\right) \to 1. \tag{3}$$

Finally, Achlioptas and C. Moore established (see [5]) the following bound for the threshold probability of 2-colorability for all sufficiently large k: there is a constant k_0 such that, for any $\varepsilon > 0$ and any fixed $k \geq k_0$,

$$\text{if } p \leq (1-\varepsilon)\,2^{k-1}\ln 2\,\frac{n}{\binom{n}{k}},\ \text{then } \mathsf{P}\left(H(n,k,p) \text{ is 2-colorable}\right) \to 1. \tag{4}$$

Together with the upper bound (2) the inequality (4) gives the exact value of the considered sharp threshold for 2-colorability in the case of fixed $k \geq k_0$:

$$p^* = 2^{k-1}\ln 2\,\frac{n}{\binom{n}{k}}.$$

The r-colorability of $H(n,k,p)$ for $r > 2$ is not studied in such detail as 2-colorability. The following lemmas are just natural generalizations of the results (1) and (2) of Alon and Spencer.

Lemma 1. *There exist positive constants $C, c > 0$ such that for any $k = k(n) \geq 3$ and $r = r(n) \geq 2$, satisfying the conditions $r^{k-1}/k \geq C$ and $r^{k-1} = o(n)$ the following statement holds:*

$$\textit{if } p \leq c\,\frac{r^{k-1}}{k^2}\,\frac{n}{\binom{n}{k}},\ \textit{then } \mathsf{P}\left(H(n,k,p) \textit{ is } r\textit{-colorable}\right) \to 1. \tag{5}$$

Lemma 2. *Let the functions $k = k(n)$ and $r = r(n)$ satisfy the relation $k^2 r = o(n)$. Then for any fixed $\varepsilon > 0$,*

$$\textit{if } p \geq (1+\varepsilon)\,r^{k-1}\ln r\,\frac{n}{\binom{n}{k}},\ \textit{then } \mathsf{P}\left(H(n,k,p) \textit{ is } r\textit{-colorable}\right) \to 0. \tag{6}$$

Another result for r-colorability of random hypergraphs was obtained by Achlioptas, Kim, Krivelevich and Tetali. In the final comment of their paper [4] they stated (with providing an algorithm of the proof) that the result (3) can be generalized to the case of r colors in the following form.

Theorem 1. (D. Achlioptas, J.H. Kim, M. Krivelevich, P. Tetali, [4]) *Suppose $k \geq 3$ and $r \geq 2$ are fixed. If $p = p(n)$ satisfies the inequality*

$$p \leq \frac{r(r+1)!}{(r+1)^{2(r+1)}} \frac{r^{k-1}}{k} \frac{n}{\binom{n}{k}}, \tag{7}$$

then $\mathsf{P}\left(H(n,k,p) \text{ is } r\text{-colorable}\right) \to 1$.

It is easy to see that the bound (5) of Lemma 1 is better than (7) if

$$k \leq c \frac{(r+1)^{2(r+1)}}{r(r+1)!} = \Omega\left(r^r\right).$$

So, Theorem 1 gives a new result only when r is small in comparison with k: $r = O(\ln k / \ln\ln k)$.

The threshold probability for r-colorability of the random hypergraph $H(n,k,p)$ in the case when r is large in comparison with k can be obtained by using the results concerning the chromatic number of $H(n,k,p)$ (recall, e.g., that (4), (7) are nontrivial only when k is much larger than r). This problem was studied by a series of researchers (see, e.g., [6], [7] for the background). In our paper we study $H(n,k,p)$ in the "sparse" case, i.e. the function $p = p(n)$ is sufficiently small. For such values of p, M. Krivelevich and B. Sudakov proved (see [7]) the following theorem.

Theorem 2. (M. Krivelevich, B. Sudakov, [7]) *Let $k \geq 3$ be fixed. There is a constant $d_0 = d_0(k)$ such that, for any $p = p(n)$ satisfying the conditions*

$$d = d(n) = (k-1)\binom{n-1}{k-1} p \geq d_0, \quad d = o(n^{k-1}),$$

the following convergence holds:

$$\mathsf{P}\left(\left(\frac{d}{k\ln d}\right)^{1/(k-1)} \leq \chi(H(n,k,p)) \leq \left(\frac{d}{k\ln d}\left(1 + \frac{28k\ln\ln d}{\ln d}\right)\right)^{1/(k-1)}\right) \to 1.$$

One can make an immediate corollary from this theorem.

Corollary 1. *Let $k \geq 3$ and $\varepsilon \in (0,1)$ be fixed. There is a constant $r_0 = r_0(k,\varepsilon)$ such that, for any $r = r(n)$ satisfying the conditions*

$$r \geq r_0, \quad r^{k-1}\ln r = o(n^{k-1}),$$

the following convergence holds:

$$\mathsf{P}\left(\chi(H(n,k,p)) \leq r\right) \to 1, \text{ where } p = (1-\varepsilon)\, r^{k-1} \ln r \, \frac{n}{\binom{n}{k}}.$$

Corollary 1 together with Lemma 2 shows that the function $p^* = r^{k-1} \ln r\, n / \binom{n}{k}$ is a threshold probability in the wide range: k is fixed, r is sufficiently large in comparison with k and $r^{k-1} \ln r = o(n^{k-1})$.

However, Theorem 2 (and, consequently, Corollary 1) can be proved not only for fixed k, but for slowly growing functions $k = k(n)$ also. The calculations from the proof of Theorem 2 provides the following necessary relations between d, p, k and n:

$$d \geq (\ln d)^{28k-27}, \quad n^{1/3} \geq \left(\ln(n^{k-1}p)\right)^{3(k-1)-1/2}. \tag{8}$$

These relations implies that in Corollary 1 we have the following restrictions:

$$r = \Omega\left(k^{29} (\ln k)^{28}\right), \quad n \geq k^{9k + O(k \ln\ln k/(\ln k))}. \tag{9}$$

So, despite the fact that Corollary 1 gives very good lower bound for the threshold probability, its statement holds only for large r in comparison with k: $r = \Omega\left(k^{29}(\ln k)^{28}\right)$. Recall that (7) is better than (5) only when $r = O(\ln k / \ln\ln k)$. Hence, in the very wide range of the values of r,

$$\frac{\ln k}{\ln\ln k} \leq r \leq k^{29} (\ln k)^{28}, \tag{10}$$

only the lower bound from Lemma 1 is known.

Remark 1. The proof of Theorem 2 seems possible to be adopted to the case of smaller values of the parameter d (and, consequently, parameter r in Corollary 1) than given by (8) . But, for example, the final condition $r > k^4$ seems to be necessary. So, the case when r is not very large in comparison with k is certainly not well studied.

We have finished discussing previously known results and now proceed to the new ones.

2 New results

Our main approach of studying the threshold for r-colorability of random hypergraph $H(n, k, p)$ is to apply methods and results concerning extremal problems of hypergraph coloring theory.

2.1 Colorings of hypergraphs with bounded vertex degrees

For all $k, r \geq 2$, let $\Delta(k, r)$ denote the minimum possible $\Delta(H)$, where H is a k-uniform non-r-colorable hypergraph. The problem of finding or estimating the value $\Delta(k, r)$ is one of the classical problems in extremal combinatorics. First bounds for $\Delta(k, r)$ were obtained by P. Erdős and L. Lovász (see [8]), they proved that for all $k, r \geq 2$,

$$\frac{r^{k-1}}{4k} \leq \Delta(k, r) \leq 20 k^2 r^{k+1}. \tag{11}$$

Kostochka and Rödl improved (see [9]) the upper bound from (11). They showed that for all $k, r \geq 2$,

$$\Delta(k, r) \leq \left\lceil k\, r^{k-1} \ln r \right\rceil.$$

Classical lower bound (11) of Erdős and Lovász was improved by J. Radhakrishnan and A. Srinivasan (see [10]) in the case $r = 2$. They proved that for large n,

$$\Delta(k,2) \geq 0.17\,\frac{2^k}{\sqrt{k\ln k}}.$$

Their result is still the best one in the case of two colors.

When $r > 2$ D.A. Shabanov proved (see [11]) a lower bound with slightly better "polynomial" factor: for any $k \geq 3$, $r \geq 3$,

$$\Delta(k,r) > \frac{1}{8}\,k^{-1/2}r^{k-1}. \tag{12}$$

The last known result concerning $\Delta(k,r)$ was recently obtained by A.V. Kostochka, M. Kumbhat and V. Rödl (see [12]). They showed that if $r = r(n) \ll \sqrt{\ln\ln k}$, then

$$\Delta(k,r) > e^{-4r^2}\left(\frac{k}{\ln k}\right)^{\frac{\lfloor\log_2 r\rfloor}{\lfloor\log_2 r\rfloor+1}}\frac{r^k}{k}. \tag{13}$$

The following lemma clarifies the connection between the value $\Delta(k,r)$ and the threshold for r-colorability of random hypergraph $H(n,k,p)$.

Lemma 3. *Suppose $k = k(n) \geq 2$ and $r = r(n) \geq 2$ satisfy the relation*

$$\frac{3}{16}\,\Delta(k,r) - \ln n \to -\infty \text{ as } n \to \infty. \tag{14}$$

If

$$p \leq \frac{1}{2}\,\frac{\Delta(k,r)}{k}\,\frac{n}{\binom{n}{k}},$$

then $\mathsf{P}\left(H(n,k,p) \text{ is } r\text{-colorable}\right) \to 1$.

Proof. Since the probability $\mathsf{P}\left(H(n,k,p) \text{ is } r\text{-colorable}\right)$ decreases with growth of p, we have to deal only with $p = \frac{1}{2}\,\frac{\Delta(k,r)}{k}\,\frac{n}{\binom{n}{k}} = \frac{1}{2}\,\Delta(k,r)\binom{n-1}{k-1}^{-1}$.

Let v be a vertex of $H(n,k,p)$ and let X_v denote the degree of v in $H(n,k,p)$. It is clear that X_v is a binomial random variable $Bin\left(\binom{n-1}{k-1},p\right)$. We shall need a classical bound on probability of large deviations for binomial variables (so called, Chernoff bound): if X is a binomial random variable, then for any $\lambda > 0$,

$$\mathsf{P}\left(X \geq \mathsf{E}X + \lambda\right) \leq \exp\left\{-\frac{\lambda^2}{2(\mathsf{E}X + \lambda/3)}\right\}. \tag{15}$$

The proof of this classical fact can be found, e.g., in the book [13]. Using (15) with $\lambda = \mathsf{E}X$ we get

$$\mathsf{P}\left(X_v \geq 2\binom{n-1}{k-1}p\right) = \mathsf{P}\left(X_v \geq \Delta(k,r)\right) \leq \exp\left\{-\frac{3}{16}\,\Delta(k,r)\right\}.$$

Consequently we obtain the following bound for the probability of the existence of the vertices with large degree in $H(n,k,p)$:

$$\mathsf{P}\left(\Delta(H(n,k,p)) \geq \Delta(k,r)\right) \leq n\,\exp\left\{-\frac{3}{16}\,\Delta(k,r)\right\} = \exp\left\{\ln n - \frac{3}{16}\,\Delta(k,r)\right\} \to 0$$

as $n \to +\infty$. The last relation follows from the condition (14). Thus, by the definition of the value $\Delta(k, r)$ we have

$$\lim_{n\to\infty} \mathsf{P}(\chi(H(n,k,p)) \leq r) \geq \lim_{n\to\infty} \mathsf{P}\left(\Delta(H(n,k,p)) < \Delta(k,r)\right) = 1.$$

Lemma 3 is proved. $\square$

As a corollary of Lemma 3 and the bounds (12) and (13) for $\Delta(k, r)$ we immediately obtain the following lower bound for the threshold for r-colorability of $H(n, k, p)$.

Corollary 2. *1) Suppose $k = k(n) \geq 3$ and $r = r(n) \geq 3$ satisfy the relation*

$$\frac{3}{128} \frac{r^{k-1}}{\sqrt{k}} - \ln n \to -\infty \text{ as } n \to \infty. \tag{16}$$

If

$$p \leq \frac{3}{32} \frac{r^{k-1}}{k^{3/2}} \frac{n}{\binom{n}{k}}, \tag{17}$$

then $\mathsf{P}\left(H(n,k,p) \text{ is } r\text{-colorable}\right) \to 1$.

2) Suppose $k = k(n) \geq 3$ and $r = r(n) \geq 2$ satisfy the relation

$$\frac{3}{16} e^{-4r^2} \left(\frac{k}{\ln k}\right)^{\frac{\lfloor \log_2 r \rfloor}{\lfloor \log_2 r \rfloor + 1}} \frac{r^k}{k} - \ln n \to -\infty \text{ as } n \to \infty.$$

If $r = o(\sqrt{\ln \ln k})$ and

$$p \leq \frac{3}{16} e^{-4r^2} \left(\frac{k}{\ln k}\right)^{\frac{\lfloor \log_2 r \rfloor}{\lfloor \log_2 r \rfloor + 1}} \frac{r^k}{k^2} \frac{n}{\binom{n}{k}}, \tag{18}$$

then $\mathsf{P}\left(H(n,k,p) \text{ is } r\text{-colorable}\right) \to 1$.

Let us compare the results of Corollary 2 with previous ones. Since Corollary 1 gives almost complete answer, we have to compare (17) and (18) with (5) from Lemma 1 and (7) from Theorem 1. Although the bounds (5) and (7) formally hold only when k and r are fixed, the analogous statements can be proved by using the same arguments for growing functions $k = k(n)$ and $r = r(n)$. For example, the analysis of the calculations in the papers [3], [4] and [5] shows that

1. The statement of Lemma 1 holds for almost all functions $k = k(n)$ and $r = r(n)$, since it is true for $r^{k-1} = o(n)$ and in the case $r^{k-1}/k \geq 22 \ln n$ we can just apply Lemma 3 with the classical bound (11) of Erdős and Lovász.

2. The proof of Theorem 1 can be extended to the following range of values of $k = k(n)$ and $r = r(n)$: r is fixed and $k = o(\sqrt{n})$.

3. The proof of the result (4) by Achlioptas and Moore does not work for any growing $k = k(n)$. It is also unclear how to generalize it to case of fixed $r > 2$. Thus, we do not compare our new results with (4), since we consider only the situation when k or r (or both of them) is a growing function of n.

Let us sum up the obtained information. The lower bound (5) from Lemma 1 holds almost for all r and k. Theorem 1 also can be extended to a wide area of the values of the parameters. Everywhere below for simplicity we shall compare only the values of the bounds.

Both (17) and (18) are obviously better than (5). The second bound (18) is worse than (7). Indeed, the right hand-side of (7) is at least

$$e^{-2r\ln r}\,\frac{r^{k-1}}{k},$$

which is better than (18), whose right hand-side is at most

$$e^{-4r^2}\,\frac{r^{k-1}}{k^{(\lfloor\log_2 r\rfloor+2)/(\lfloor\log_2 r\rfloor+1)}}.$$

The first bound (17) of Corollary 2 is better than (7) if

$$\sqrt{k}<\frac{3}{32}\,\frac{(r+1)^{2(r+1)}}{r(r+1)!}.$$

This inequality holds, e.g., when $r\geq \ln k/\ln\ln k$.

Let us make intermediate conclusions. Our new lower bound (7) for the threshold probability of r-colorability of random hypergraph $H(n,k,p)$ improves all previously known results in the following wide area (see condition (9) of Corollary 1 and condition (16) of Corollary 2):

$$\ln k/\ln\ln k\leq r\leq k^{29}(\ln k)^{28}\quad\text{ and }\quad\frac{r^{k-1}}{\sqrt{k}}\gg\ln n. \tag{19}$$

We see that in the area (19) the parameter r cannot be very small in comparison with the number of vertices n, but it can be very large. In the next paragraph we shall present a better bound when r is not very small and also is not very large in comparison with n.

2.2 Main result

The main result of our paper is formulated in the following theorem.

Theorem 3. *Suppose $\delta\in(0,1)$ is a constant. Let $k=k(n)$ and $r=r(n)\geq 2$ satisfy the following conditions: $k\geq k_0(\delta)$, where $k_0(\delta)$ is some constant, and, moreover,*

$$(k-1)\ln r<\frac{1-\delta}{2}\ln n,\quad r^{k-1}k^{-\varphi(k)}\geq 6\ln n, \tag{20}$$

where $\varphi(k)=4\left\lfloor\sqrt{\frac{\ln k}{\ln(2\ln k)}}\right\rfloor^{-1}$. Then for function $p=p(n)$, satisfying

$$p\leq\frac{1}{2}\,\frac{r^{k-1}}{k^{1+\varphi(k)}}\,\frac{n}{\binom{n}{k}}, \tag{21}$$

we have $\mathsf{P}\left(H(n,k,p)\text{ is } r\text{-colorable}\right)\to 1$ as $n\to\infty$.

Let us compare the result of Theorem 3 with previous ones. It is clear that the restriction (21) is weaker (for all sufficiently large k) than our previous results (17) and (18) obtained in §2.1. It is also obvious that (21) is better than the lower bound (5) from Lemma 1 for all sufficiently large k. So, it remains only to compare (21) with the result of Theorem 1 proved by Achlioptas, Krivelevich, Kim and Tetali. We see that (21) is asymptotically better than (7) if

$$k^{4\left\lfloor\sqrt{\frac{\ln k}{\ln(2\ln k)}}\right\rfloor^{-1}} < \frac{(r+1)^{2(r+1)}}{2r(r+1)!}.$$

This inequality holds, e.g., in the following asymptotic area: $r \gg \sqrt{\ln k}$.

Thus, our main result (21) gives a new lower bound for the threshold probability for r-colorability of the random hypergraph $H(n,k,p)$. This new bound improves all the previously known results in the wide area of the parameters (recall that we are working in the area (10)):

$$\sqrt{\ln k} \ll r \leq k^{29}(\ln k)^{28} \text{ and } 6\,k^{\varphi(k)}\ln n \leq r^{k-1} \leq n^{(1-\delta)/2}.$$

For example, (21) provides a new bound when $k \sim r \sim \ln n/(5\ln\ln n)$. Moreover, our result (21) is only $k^{1+\varphi(k)}\ln r$ times smaller than the upper bound (6) from Lemma 2.

The proof of Theorem 3 is based on some result concerning colorings of 2-simple hypergraphs with bounded vertex degrees. The study of problems for colorings of simple hypergraphs was initiated by Erdős and Lovász in [8]. Later the extremal problems concerning colorings of l-simple hypergraphs with bounded vertex degrees have been considered by Z. Szabó (see [14]), A.V. Kostochka and M. Kumbhat (see [15]), D.A. Shabanov (see [16]). To prove Theorem 3 we consider 2-simple hypergraphs with a few 3-cycles.

Let H be an arbitrary hypergraph with the following properties: H is k-uniform, $\chi(H) > r$, H is 2-simple and for every edge of H, there are at most ω 3-cycles containing that edge. The class of all such hypergraphs we will denote by $\mathcal{H}(k,r,\omega)$. Let us consider the following extremal value:

$$\Delta(\mathcal{H}(k,r,\omega)) = \min\left\{\Delta(H):\ H \in \mathcal{H}(k,r,\omega)\right\}.$$

Theorem 4 gives an asymptotic lower bound for $\Delta(\mathcal{H}(k,r,\omega))$ for $\omega = \lfloor\sqrt{\ln k/(\ln\ln k)}\rfloor$.

Theorem 4. *There exists an integer k_0 such that for all $k \geq k_0$, all $r \geq 2$ and all $\omega \leq \sqrt{\ln k/(\ln\ln k)}$,*

$$\Delta(\mathcal{H}(k,r,\omega)) > r^{k-1}k^{-4\left\lfloor\sqrt{\frac{\ln k}{\ln(2\ln k)}}\right\rfloor^{-1}}. \tag{22}$$

It should be noted that the inequality (22) holds for all possible values of the parameter r, which is important for studying r-colorability of random hypergraphs. For a k-uniform, 2-simple, non-r-colorable hypergraph the lower bound for the maximum vertex degree similar to (22) is known only in the case of small r in comparison with k: $r = O(\ln k)$ (see [16] for the details).

The structure of the rest of the article will be the following. In the next paragraph we shall deduce Theorem 3 from Theorem 4. Section 3 will be devoted to the proof of Theorem 4. Finally, in Section 4 we shall discuss choosability in random hypergraphs.

2.3 Proof of Theorem 3

Due to the decreasing of the probability $\mathsf{P}\,(H(n,k,p)$ is r-colorable) with growth of p we have to deal only with $p = \frac{r^{k-1}}{2k^{1+\varphi(k)}} \frac{n}{\binom{n}{k}}$. We want to apply Theorem 4 to random hypergraph $H(n,k,p)$, so, we have to show that with probability tending to 1 $H(n,k,p)$ satisfies the following conditions: it is 2-simple, every edge is contained in at most $\omega = \lfloor\sqrt{\ln k/\ln\ln k}\rfloor$ 3-cycles and, moreover, $\Delta(H(n,k,p)) < \Delta(\mathcal{H}(k,r,\omega))$.

Let v be a vertex of the random hypergraph $H(n,k,p)$ and let X_v denote the degree of v in $H(n,k,p)$. It is clear that X_v is a binomial random variable $Bin\left(\binom{n-1}{k-1},p\right)$. Using Chernoff bound (15) with $\lambda = \mathsf{E}X_v$ and the condition (20), we have

$$\mathsf{P}\left(X_v \geq r^{k-1}k^{-\varphi(k)}\right) \leq \exp\left\{-3r^{k-1}k^{-\varphi(k)}/16\right\} \leq \exp\{-(9/8)\ln n\} = n^{-9/8}.$$

Thus,

$$\mathsf{P}\left(\Delta(H(n,k,p)) \geq r^{k-1}k^{-\varphi(k)}\right) \leq n\cdot n^{-9/8} = o(1). \tag{23}$$

Let Y denote the number of pairs of edges, whose intersection has cardinality at least 3, and let Z denote the number of edges, which are contained in a large number, more than ω, of 3-cycles. We estimate the expectations of these two random variables:

$$\mathsf{E}Y \leq \binom{n}{3}\binom{n-3}{k-3}^2 p^2 \leq \frac{n^3k^6}{n^6}\binom{n}{k}^2\left(\frac{r^{k-1}}{k}n\binom{n}{k}^{-1}\right)^2 = \frac{\left(r^{k-1}\right)^2k^4}{n}.$$

Consequently,

$$\ln \mathsf{E}Y = (2k-2)\ln r + 4\ln k - \ln n \overset{(20)}{\leq} (1-\delta)\ln n + 4\ln k - \ln n \leq -\frac{\delta}{2}\ln n.$$

Hence, $\lim_{n\to\infty}\mathsf{E}Y = 0$ and

$$\mathsf{P}\,(H(n,k,p) \text{ is 2-simple}) \to 1. \tag{24}$$

Now we will consider edges, that are contained in a large number of triangles. Suppose u is an edge of $H(n,k,p)$. Let us denote by T_u the set of all triangles, containing u. Furthermore, we denote by $D(u',u)$ *the degree of an edge* u' *with respect to* u, a number of 3-cycles from T_u, containing an edge $u' \neq u$. Similarly, for any vertex $v \in V(H(n,k,p))$, we denote by $d(v,u)$ the *the degree of vertex* v *with respect to* u, a number of triangles (u,u',u'') from T_u such that $v \in (u' \cap u'')\setminus u$.

Now we will estimate the number of edges that have big degree with respect to T_u for some u. Denote by $Z_1(u)$ the number of edges, having degree greater than 4 with respect to u, i.e.

$$Z_1(u) = \left|\{u' \in E(H(n,k,p)) :\ D(u',u) > 4\}\right|.$$

Moreover, let us denote $Z_1 = \sum\limits_{e\in\ E(H(n,k,p))} Z_1(u)$. Now we estimate the expectation of Z_1:

$$\mathsf{E}Z_1 \leq n\binom{n-1}{k-1}^2 k^{10}\binom{n-2}{k-2}^5 p^7 \leq \binom{n}{k}^7\frac{k^{22}}{n^{11}}\left(\frac{r^{k-1}}{k}n\binom{n}{k}^{-1}\right)^7 = \frac{r^{7(k-1)}k^{15}}{n^4} =$$

$$= \exp\{7(k-1)\ln r + 15\ln k - 4\ln n\} \overset{(20)}{\leq} \exp\left\{\frac{7}{2}(1-\delta)\ln n + 15\ln k - 4\ln n\right\} \leq n^{-1/2}.$$

Let us explain the first inequality. At first we choose the vertex from the intersection of the edge u' with large degree and the edge u. Then we choose the rest vertices of these two edges. Then we choose 5 vertices on both of edges, that correspond to remaining vertices of five 3-cycles. Then we choose the last edge of each 3-cycle.

Thus,

$$\mathsf{P}\left(\text{for any } u, u' \in E(H(n,k,p)), \ \ D(u',u) \leq 4\right) \to 1. \tag{25}$$

Similarly, we shall show, that with probability tending to one, $d(v,u) \leq 4$ for any edge u and any vertex $v \notin u$. Namely, we denote by Z_2 the number of pairs v, u, such that $d(v,u) \geq 5$. Then the expectation of Z_2 can be estimated from above as follows.

$$\mathsf{E}Z_2 \leq n\binom{n-1}{k} k^5 \binom{n-2}{k-2}^5 p^6 \leq \binom{n}{k}^6 \frac{k^{15}}{n^9}\left(\frac{r^{k-1}}{k} n \binom{n}{k}^{-1}\right)^6 = \frac{r^{6(k-1)}k^9}{n^3} =$$

$$= \exp\{6(k-1)\ln r + 9\ln k - 3\ln n\} \overset{(20)}{\leq} \exp\{3(1-\delta)\ln n + 9\ln k - 3\ln n\} \leq n^{-\delta/2}.$$

So,

$$\mathsf{P}\left(\text{for any vertex } v \text{ and an edge } u \in E(H(n,k,p)), \ \ d(v,u) \leq 4\right) \to 1. \tag{26}$$

Let us introduce the following event

$$\mathcal{A}_n = \left\{\text{for any } u, u' \in E(H(n,k,p)) \text{ and any } v \in V(H(n,k,p)), \ D(u',u) \leq 4 \text{ and } d(v,u) \leq 4\right\}.$$

Due to (25) and (26) we have that $\mathsf{P}(\mathcal{A}_n) \to 1$.

Now suppose that the event $\mathcal{A}_n$ holds and there is an edge u in $H(n,k,p)$, which is contained in at least ω 3-cycles. Consider the following set of vertices V_u:

$$V_u = \{v \in V : \ \ d(v,u) > 0\}.$$

For any $v \in V_u$, by $E(v,u)$ we denote the set of edges, containing v, which also belongs to one of the 3-cycles from T_u. The event $\mathcal{A}_n$ implies that $d(v,u) \leq 4$ for any v and u and, hence, $|E(v,u)| \leq 8$ and $|V_u| \geq \frac{1}{8}|T_u| = \frac{1}{8}\omega$. Now we will construct a some convenient subset of T_u of sufficient size.

First, we have $V_u^0 = V_u$ and $T_u^0 = T_u$. Suppose sets V_u^s and T_u^s are considered. We form a set V_u^{s+1} and a set T_u^{s+1} by the following way. We choose an arbitrary 3-cycle $t_{s+1} = (u_s^{'}, u_s^{''}, u) \in T_u^s$ and an arbitrary $v_{s+1} \in (u_s^{'} \cap u_s^{''}) \setminus u$. Then we take

$$V_u^{s+1} = V_u^s \setminus \{v_{s+1}\},$$

$$T_u^{s+1} = T_u^s \setminus \{t \in T_u^s : \ t \text{ contains an edge from } E(v_{s+1}, u)\}.$$

Then we repeat the same procedure with sets T_u^{s+1}, V_u^{s+1}. The procedure continues until both sets V_u^{s+1} and T_u^{s+1} are not empty.

How many steps of procedure can we guarantee? Since the event $\mathcal{A}_n$ holds, we have that$|T_u^{i+1}| \geq |T_u^i| - 32$. Indeed, $E(v_{i+1}, u)$ consists of at most 8 edges and every edge belongs to at most 4 3-cycles. So, we can guarantee at least $\omega' = \lceil \frac{1}{32}\omega \rceil$ steps of the procedure.

Consider the obtained set of 3-cycles $\{t_1, \ldots, t_{\omega'}\}$, $t_s = (u_s^{'}, u_s^{''}, u)$, $s = 1, \ldots, \omega'$ and the set of vertices $\{v_1, \ldots, v_{\omega'}\}$. Our procedure shows that all the edges $u_1^{'}, u_1^{''}, u_2^{'}, u_2^{''}, \ldots, u_{\omega'}^{'}, u_{\omega'}^{''}$ are distinct, all the vertices $v_1, \ldots, v_{\omega'}$ are also distinct and, for any $s = 1, \ldots, \omega'$, we have $v_s \in u_s^{'} \cap u_s^{''} \setminus u$. Let us estimate the probability of the event (denoted by $\mathcal{B}_n$) that, for some edge u, the described above configuration of appears in $H(n, k, p)$. It is clear that

$$\mathsf{P}(\mathcal{B}_n) \leq \binom{n}{k} k^{2\omega'} (n-k)^{\omega'} \binom{n-2}{k-2}^{2\omega'} p^{2\omega'+1} \leq \frac{n^{\omega'} k^{6\omega'}}{n^{4\omega'}} \binom{n}{k}^{2\omega'+1} \left(\frac{r^{k-1}}{k} n \binom{n}{k}^{-1}\right)^{2\omega'+1} =$$

$$= \frac{\left(r^{k-1}\right)^{2\omega'+1} k^{4\omega'-1}}{n^{\omega'-1}}.$$

Hence, for all $k \geq k_0$,

$$\ln \mathsf{P}(\mathcal{B}_n) = (k-1)(2\omega'+1)\ln r + (4\omega'-1)\ln k - (\omega'-1)\ln n \overset{(20)}{\leq}$$

$$\overset{(20)}{\leq} (\omega'-1)\left((1-\delta)\left(\frac{2\omega'+1}{2\omega'-2}\right)\ln n + \frac{4\omega'-1}{(\omega'-1)}\ln k - \ln n\right) \leq (\omega'-1)\left(-\frac{\delta}{2}\ln n\right). \quad (27)$$

Consequently, $\lim_{n\to\infty} \mathsf{P}(\mathcal{B}_n) = 0$. Finally, if $\mathcal{A}_n$ holds, then the event that there is an edge $u \in E(H(n,k,p))$ with $|T_u| > \omega$ implies the evnt $\mathcal{B}_n$. Thus,

$$\mathsf{P}(\text{there is } u \in E(H(n,k,p)) : \ |T_u| > \omega) \leq \mathsf{P}\left(\overline{\mathcal{A}_n}\right) + \mathsf{P}(\mathcal{B}_n) \to 0. \quad (28)$$

Let us sum up the obtained information. It follows from (24) and (28) that with probability tending to 1 the random hypergraph $H(n,k,p)$ satisfies all the conditions of being an element of $\mathcal{H}(k, r, \omega(k))$, except the condition $\chi(H(n,k,p)) > r$. Applying Theorem 4 and (23) we get that $H(n,k,p) \notin \mathcal{H}(k,r,\omega(k))$ with high probability. Thus, $\mathsf{P}(H(n,k,p)$ is r-colorable$) \to 1$ as $n \to \infty$. Theorem 3 is proved.

Remark 2. If $k = k(n) \to +\infty$ then the parameter $\delta \in (0,1)$ from Theorem 3 can be taken equal to some infinitesimal function. For example, it follows from (27) that

$$\delta = 50\sqrt{\frac{\ln \ln k}{\ln k}}$$

is sufficient.

3 Proof of Theorem 4

The proof of Theorem 4 is based on the method of random recoloring. This method in the case of two colors was developed in the papers of J. Beck [17], J. Spencer [18], Radhakrishnan and Srinivasan [10]. In this paper we follow the work [16] concerning r-colorings of l-simple hypergraphs with bounded edge degrees.

The structure of the proof will be the following. In the next section we will formulate a multiparametric Theorem 5 which provides a new lower bound of the maximum edge degree in a hypergraph from the class $\mathcal{H}(k, r, \omega)$. Then we will prove Theorem 5. Finally we deduce Theorem 4 from Theorem 5 by choosing the values of the required parameters.

3.1 General theorem

Theorem 4 is a simple corollary of the following multiparametric theorem.

Theorem 5. *Let $k \geq 3$, $r \geq 2$, $\omega \geq 1$ be integers, let b, α be positive numbers. Let us denote:*

$$t = \left\lfloor \sqrt{\frac{\ln k}{\ln(\alpha \ln k)}} \right\rfloor, \quad q = \frac{\alpha \ln k}{k}. \tag{29}$$

Let $H = (V, E)$ be an k-uniform 2-simple hypergraph such that for every edge in H there are at most ω 3-cycles that contain that edge. Let, moreover, every edge of hypergraph H intersects at most d other edges of H, where

$$d \leq r^{k-1} k^{1-b/t} - 1. \tag{30}$$

If the following inequalities hold

$$b \leq t < k - \omega, \tag{31}$$

$$\frac{2}{k} \leq q \leq \frac{1}{2}, \tag{32}$$

$$\frac{k^2}{2^k} + (t+1)k^{1-\alpha} e^{\alpha(\ln k)(t+\omega)/k} (\alpha \ln k)^{t+\omega} + \frac{(t+1)^2}{t!} k^{2-b} (\alpha \ln k)^{t\omega} + \\ + (t+1)t \left(\frac{2e\alpha \ln k}{t-1}\right)^{t-1} k^{1+\alpha-b} < \frac{1}{4} \tag{33}$$

then $\chi(H) \leq r$.

The proof of this theorem is based on a method of vertex random coloring. To prove Theorem 5 we have to show the existence of a proper vertex r-coloring for hypergraph H. We shall construct some random r-coloring and estimate the probability that this coloring is not proper for H. If this probability is greater than 0, then we prove the existence of a required coloring, and the theorem follows.

3.2 Algorithm of random recoloring

We follow the ideas of Radhakrishnan and Srinivasan from [10] and the construction from [16] concerning random recoloring. Let $V = \{v_1, \ldots, v_w\}$. The algorithm consists of two phases.

Phase 1. We color all vertices randomly and uniformly with r colors, independently from each other. Let us denote the generated random coloring by χ_0.

The obtained coloring χ_0 can contain monochromatic edges and "almost monochromatic" edges. An edge $e \in E$ is said to be *almost monochromatic* in χ_0 if there is a color u such that

$$n - t - \omega + 2 \leq |\{v \in e : \ v \text{ is colored by } u \text{ in } \chi_0\}| < n.$$

In this case, the color u is called *dominating* in e. For every $v \in V$, $u = 1, \ldots, r$, let us use the notations

$$\mathcal{M}(v) = \{e \in E : \ v \in e, \ e \text{ is monochromatic in } \chi_0\},$$

$$\mathcal{AM}(v, u) = \{e \in E : \ v \in e, \ e \text{ is almost monochromatic in } \chi_0 \text{ with dominating color } u\}.$$

Phase 2. In this phase, we want to recolor some vertices from the edges, which are monochromatic in χ_0. We consider the vertices according to an arbitrary fixed order $v_1, \ldots, v_w$. Let $\{\eta_1, \ldots, \eta_w\}$ be mutually independent equally distributed random variables, taking values $1, \ldots, r$ with the same probability p (the value of the parameter p will be chosen later) and the value 0 with probability $1 - rp$. The recoloring procedure consists of w steps.

Step 1. Assume that $\mathcal{M}(v_1) \neq \emptyset$ and, moreover, there is no $u = 1, \ldots, r$ and $e \in \mathcal{AM}(v_1, u)$ such that

(a) $\eta_1 = u$,

(b) v_1 is the only vertex in e, which is not colored by u in χ_0.

Then we try to recolor v_1 according to the value of the random variable η_1:

- if $\eta_1 = 0$, then we do not recolor v_1,
- if $\eta_1 \neq 0$, then we recolor v_1 in the color η_1.

In all the other situations, we do not change the color of v_1. Let χ_1 be the coloring after considering v_1.

Now let the vertices $v_1, \ldots, v_{i-1}$ have been considered, so that the coloring χ_{i-1} is obtained.

Step i. Assume that some $f \in \mathcal{M}(v_i)$ is still monochromatic in χ_{i-1} and, moreover, there is no $u = 1, \ldots, r$ and $e \in \mathcal{AM}(v_i, u)$ such that

(a) $\eta_i = u$,

(b) v_i is the only vertex in e, which is not colored by u in χ_{i-1}.

Then we try to recolor v_i according to the value of the random variable η_i:

- if $\eta_i = 0$, then we do not recolor v_i,
- if $\eta_i \neq 0$, then we recolor v_i in the color η_i.

In all the other situations, we do not change the color of v_i. Let the resulting coloring be χ_i.

Let $\tilde{\chi}$ be the coloring obtained after the consideration of all the vertices.

Now we are going to give a more formal construction of the random coloring $\tilde{\chi}$ using the techniques of random variables. This is very useful for the further proof. We analyze the event $\mathcal{F}$ that $\tilde{\chi}$ is not a proper coloring for H. We divide $\mathcal{F}$ into some "local" events and estimate their probabilities. Finally, we use Local Lemma to show that all these events do not occur simultaneously with positive probability. This implies that $\tilde{\chi}$ is a proper coloring of H with positive probability, and, hence, H is r-colorable.

3.3 Formal Construction of the random coloring from §3.2

Without loss of generality, we may assume, that $V = \{1, 2, 3, \ldots, w\}$. Let us also fix an arbitrary ordering of the edges of H. Consider, on some probability space, the following set of mutually independent random elements:

1. $\xi_1, \ldots, \xi_w$ — equally distributed random variables, taking values $1, 2, \ldots, r$ with equal probability $1/r$.

2. $\eta_1, \ldots, \eta_w$ — equally distributed random variables taking values $1, 2, \ldots, r$ with equal probability p and the value 0 with probability $1 - rp$. We take the parameter p equal to $p = q/(r-1)$. By the condition (32) such choice of the parameter is correct, i. e., for every $r \geq 2$, one has the inequalities $rp \leq r/(2(r-1)) \leq 1$.

Let $e \in E$ be an edge of H. For every $u = 1, \ldots, r$, let $\mathcal{M}(e, u)$ and $\mathcal{AM}(e, u)$ denote the following events:

$$\mathcal{M}(e,u) = \bigcap_{s \in e} \{\xi_s = u\}, \quad \mathcal{AM}(e,u) = \left\{ 0 < \sum_{s \in e} I\{\xi_s \neq u\} \leq t + \omega - 2 \right\}. \tag{34}$$

We shall introduce successively random variables ζ_i, $i = 1, \ldots, w$. Let $\mathcal{D}_1$ and $\mathcal{A}_1$ denote the following events:

$$\mathcal{D}_1 = \bigcup_{e \in E\colon 1 \in e} \bigcup_{u=1}^{r} \mathcal{M}(e, u),$$

$$\mathcal{A}_1 = \bigcup_{f \in E\colon 1 \in f} \bigcup_{u=1}^{r} \left(\left\{ \xi_1 \neq u,\ \eta_1 = u,\ \sum_{s \in f\colon s > 1} I\{\xi_s = u\} = k - 1 \right\} \cap \mathcal{AM}(f, u) \right),$$

and let

$$\zeta_1 = \xi_1\, I\{\overline{\mathcal{D}_1} \cup \{\eta_1 = 0\} \cup \mathcal{A}_1\} + \eta_1\, I\{\mathcal{D}_1 \cap \{\eta_1 \neq 0\} \cap \overline{\mathcal{A}_1}\}.$$

For every $i > 1$, let $\mathcal{D}_i$ and $\mathcal{A}_i$ denote the events

$$\mathcal{D}_i = \bigcup_{e \in E\colon i \in e} \bigcup_{u=1}^{r} \left\{ \mathcal{M}(e, u) \cap \bigcap_{s \in e\colon s < i} \{\zeta_s = u\} \right\},$$

$$\mathcal{A}_i = \bigcup_{f \in E\colon i \in f} \bigcup_{u=1}^{r} \left(\left\{ \xi_i \neq u,\ \eta_i = u,\ \sum_{s \in f\colon s < i} I\{\zeta_s = u\} + \sum_{s \in f\colon s > i} I\{\xi_s = u\} = k - 1 \right\} \cap \mathcal{AM}(f, u) \right).$$

We define ζ_i in the following way:

$$\zeta_i = \xi_i\, I\{\overline{\mathcal{D}_i} \cup \{\eta_i = 0\} \cup \mathcal{A}_i\} + \eta_i\, I\{\mathcal{D}_i \cap \{\eta_i \neq 0\} \cap \overline{\mathcal{A}_i}\}.$$

It is easy to see that the random variables ζ_i take values only from $\{1, 2, \ldots, r\}$. So, we may interpret the random vector $\vec{\zeta} = (\zeta_1, \ldots, \zeta_w)$ as a random r-coloring of the vertex set V (we assign the color ζ_i to the vertex i). Let $\mathcal{F}$ denote the event that $\vec{\zeta}$ is not a proper coloring of the hypergraph H, i. e.,

$$\mathcal{F} = \bigcup_{e \in E} \bigcup_{u=1}^{r} \bigcap_{s \in e} \{\zeta_s = u\}. \tag{35}$$

Our task is to prove that $\mathsf{P}(\mathcal{F}) < 1$ under the conditions of Theorem 5.

We shall divide the event $\bigcap_{s\in e}\{\zeta_s = u\}$ into three parts, depending on the behavior of the random variables $\{\xi_s:\ s \in e\}$. Let $\mathcal{C}_0(e,u)$, $\mathcal{C}_1(e,u)$, $\mathcal{C}_2(e,u)$ be the following events:

$$\mathcal{C}_0(e,u) = \bigcup_{a=1,\, a\neq u}^{r} \bigcap_{s\in e}\{\zeta_s = u,\ \xi_s = a\}, \quad \mathcal{C}_1(e,u) = \bigcap_{s\in e}\{\zeta_s = u,\ \xi_s = u\},$$

$$\mathcal{C}_2(e,u) = \bigcap_{s\in e}\{\zeta_s = u\} \cap \bigcap_{a=1}^{r} \overline{\mathcal{M}(e,a)}. \tag{36}$$

We shall consider these events separately. But before we establish a simple inequality, which we will use later. It follows from (32) that

$$\alpha \ln k = qk \geq 2. \tag{37}$$

Note that the last inequality implies that the parameter t in (29) is correctly defined (there is no negative number under the square root).

3.4 First part of $\mathcal{F}$: the event $\mathcal{C}_0(e,u)$

If the event $\mathcal{C}_0(e,u)$ occurs, then for every $s \in e$, one has $\zeta_s = \eta_s$, since $\zeta_s \neq \xi_s$. We get the relation

$$\bigcup_{u=1}^{r} \mathcal{C}_0(e,u) \subset \bigcup_{u=1}^{r} \bigcup_{a=1,\, a\neq u}^{r} \bigcap_{s\in e}\{\eta_s = u,\ \xi_s = a\} = \mathcal{Q}_0(e). \tag{38}$$

The probability of the event $\mathcal{Q}_0(e)$ can be easily calculated:

$$\mathsf{P}(\mathcal{Q}_0(e)) = \sum_{u=1}^{r} \sum_{a=1,\, a\neq u}^{r} \prod_{s\in e} \mathsf{P}\left(\eta_s = u,\ \xi_s = a\right) = r(r-1)\left(\frac{p}{r}\right)^k. \tag{39}$$

3.5 Second part of $\mathcal{F}$: the event $\mathcal{C}_1(e,u)$

Suppose that the event $\mathcal{C}_1(e,u)$ occurs. This event, obviously, implies all the events $\mathcal{D}_s,\ s \in e$. Then the equality $\xi_s = \zeta_s = u$ for a vertex $s \in e$ can happen in two ways: either $\eta_s \in \{0,u\}$, or $\eta_s \notin \{0,u\}$ and the event $\mathcal{A}_s$ occurs. Consider the following partition of the event $\mathcal{C}_1(e,u)$:

$$\mathcal{C}_1(e,u) = \mathcal{S}_0(e,u) \cup \mathcal{S}_1(e,u), \tag{40}$$

where

$$\mathcal{S}_0(e,u) = \mathcal{C}_1(e,u) \cap \left\{\sum_{s\in e} I\left\{\eta_s \notin \{0,u\}\right\} \leq t + \omega - 1\right\},$$

$$\mathcal{S}_1(e,u) = \mathcal{C}_1(e,u) \cap \left\{\sum_{s\in e} I\left\{\eta_s \notin \{0,u\}\right\} > t + \omega - 1\right\}.$$

Consider the event $\mathcal{S}_0(e,u)$. By the definition (36) of the event $\mathcal{C}_1(e,u)$ the following relation holds:

$$\mathcal{S}_0(e,u) \subset \bigcap_{s\in e}\{\xi_s = u\} \cap \left\{\sum_{s\in e} I\left\{\eta_s \notin \{0,u\}\right\} \leq t + \omega - 1\right\}.$$

Let $\mathcal{Q}_1(e)$ denote the union of the last events:

$$\bigcup_{u=1}^{r} \mathcal{S}_0(e,u) \subset \bigcup_{u=1}^{r} \left\{ \bigcap_{s\in e} \{\xi_s = u\} \cap \left\{ \sum_{s\in e} I\left\{\eta_s \notin \{0,u\}\right\} \leq t+\omega-1 \right\} \right\} = \mathcal{Q}_1(e). \tag{41}$$

The probability of $\mathcal{Q}_1(e)$ has the following estimate:

$$\mathsf{P}\left(\mathcal{Q}_1(e)\right) = r^{1-k} \sum_{j=0}^{t+\omega-1} \binom{k}{j} q^j (1-q)^{k-j} \leq r^{1-k}(1-q)^{k-t-\omega} \sum_{j=0}^{t+\omega-1} (kq)^j \leq$$

$$\leq r^{1-k}(1-q)^{k-t-\omega}(kq)^{t+\omega}. \tag{42}$$

The last inequality follows from the bound (37): $kq = \alpha \ln k \geq 2$.

Consider now the event $\mathcal{S}_1(e,u)$. Let us fix $v \in e$ satisfying $\eta_v \notin \{0,u\}$. As it was noted above, the event $\mathcal{A}_v$ should happen for every such vertex. This event implies that for some edge f, $v \in f$, and some color $a \neq u$, the following event has to occur

$$\mathcal{W}(v,f,u,a) = \left\{ \xi_v = u,\ \eta_v = a,\ \sum_{s\in f:\, s<v} I\{\zeta_s = a\} + \sum_{s\in f:\, s>v} I\{\xi_s = a\} = k-1 \right\} \cap \mathcal{AM}(f,a).$$

It is easy to show that $f \neq e$, moreover, $f \cap e = \{v\}$. Indeed, for all $s \in e$, it holds that $\xi_s = \zeta_s = u$, but for all $s \in f \backslash \{v\}$, either $\zeta_s = a$, or $\xi_s = a$.

Suppose $\{v_1,\ldots,v_h\} = \{v \in e:\ \ \eta_v \notin \{0,u\}\}$. For any $i = 1,\ldots,h$, the event $\mathcal{S}_1(e,u)$ implies the event $\mathcal{W}(v_i,f_i,u,a_i)$ for some edge f_i satisfying $\{v_i\} = f_i \cap e$ and some color $a_i \neq u$. Moreover, $\mathcal{S}_1(e,u)$ also implies that $h = h(e,u) \geq t+\omega$. Since there are at most ω 3-cycles containing e, there is a subset $\{f_1',\ldots,f_t'\} \subset \{f_1,\ldots,f_h\}$ such that f_i' and f_j' are disjoint for all $i \neq j$.

For further convenience, we introduce a notation of *the configuration of the first type*. For given edge e, the set of edges $\{f_1,\ldots,f_t\}$ is said to be the configuration of the first type (denotation: $\{f_1,\ldots,f_t\} \in \mathrm{CONF1}(e)$) if, for any any $i = 1,\ldots,t$, $|f_i \cap e| = 1$ and, moreover, all the edges f_i are pairwise disjoint.

Thus, by the above arguments we the following relation

$$\mathcal{S}_1(e,u) \subset \bigcap_{s\in e}\{\xi_s = u\} \cap \bigcup_{\substack{a_1,\ldots,a_t=1\\ a_i\neq u}}^{r} \ \bigcup_{\{f_1,\ldots,f_t\}\in \mathrm{CONF1}(e)} \ \bigcap_{i=1}^{t} \mathcal{W}(e\cap f_i, f_i, u, a_i), \tag{43}$$

where the set of edges $\{f_1,\ldots,f_t\}$ is assumed to be ordered according to the originally selected ordering of E, i. e. the number of the edge f_i is less than the number of the edge f_j, if $i<j$. Let us use the notations: $\widehat{f_i} = f_i \backslash e$ and $v_i = e \cap f_i$, $i = 1,\ldots,t$. It follows from the definition of the configuration of the first type that the sets $\widehat{f_i}$, $i = 1,\ldots,t$ do not have common vertices, i.e. $\widehat{f_i} \cap \widehat{f_j} = \emptyset$, if $i \neq j$. Furthermore, $|\widehat{f_i}| = k-1$.

If the event $\mathcal{W}(e\cap f_i, f_i, u, a_i)$ happens, then by $\mathcal{AM}(f_i,a_i)$ the edge f_i contains at most $t+\omega-2$ vertices s, satisfying $\xi_s \neq a_i$. Moreover, for all such vertices, $\zeta_s = a_i$, and so, $\zeta_s = \eta_s = a_i$.

The set $\widehat{f_i}$ contains at most $t+\omega-3$ such vertices, since the vertex v_i doesn't belong to $\widehat{f_i}$ and $\xi_{v_i} = u \neq a_i$. Thus, we obtain the relation

$$\bigcap_{s\in e}\{\xi_s = u\} \cap \bigcap_{i=1}^{t} \mathcal{W}(e\cap f_i, f_i, u, a_i) \subset \bigcap_{s\in e}\{\xi_s = u\} \cap \bigcap_{i=1}^{t}\{\eta_{v_i} = a_i\}\cap$$

$$\cap \bigcap_{i=1}^{t}\left\{\bigcap_{s\in \widehat{f_i}}\left(\{\xi_s \neq a_i,\ \eta_s = a_i\} \cup \{\xi_s = a_i\}\right)\right\} \cap \bigcap_{i=1}^{t}\left\{\sum_{s\in \widehat{f_i}} I\{\xi_s \neq a_i\} \leq t+\omega-3\right\}. \tag{44}$$

Let $\mathcal{Q}_2(e,F)$ denote the union of the last events, where $F = \{f_1,\ldots,f_t\} \in \mathrm{CONF1}(e)$:

$$\mathcal{Q}_2(e,F) = \bigcup_{u=1}^{r} \bigcup_{\substack{a_1,\ldots,a_t=1\\ a_i\neq u}}^{r} \left\{\bigcap_{s\in e}\{\xi_s = u\} \cap \bigcap_{i=1}^{t}\{\eta_{v_i} = a_i\}\cap\right.$$

$$\left.\cap \bigcap_{i=1}^{t}\left\{\bigcap_{s\in \widehat{f_i}}\left(\{\xi_s \neq a_i,\ \eta_s = a_i\} \cup \{\xi_s = a_i\}\right)\right\} \cap \bigcap_{i=1}^{t}\left\{\sum_{s\in \widehat{f_i}} I\{\xi_s \neq a_i\} \leq t+\omega-3\right\}\right\}. \tag{45}$$

The relations (43) and (44) imply

$$\bigcup_{u=1}^{r} \mathcal{S}_1(e,u) \subset \bigcup_{F\in\, \mathrm{CONF1}(e)} \mathcal{Q}_2(e,F). \tag{46}$$

Let us estimate the probability of $\mathcal{Q}_2(e,F)$:

$$\mathsf{P}\left(\mathcal{Q}_2(e,F)\right) = \sum_{u=1}^{r} \sum_{\substack{a_1,\ldots,a_t=1\\ a_i\neq u}}^{r} r^{-k}p^t \prod_{i=1}^{t} \sum_{j=0}^{t+\omega-3} \binom{|\widehat{f_i}|}{j} \left(\frac{r-1}{r}\right)^j p^j \left(\frac{1}{r}\right)^{|\widehat{f_i}|-j} =$$

$$= r(r-1)^t r^{-k} p^t r^{-\sum\limits_{i=1}^{t}|\widehat{f_i}|} \prod_{i=1}^{t} \sum_{j=0}^{t+\omega-3} \binom{|\widehat{f_i}|}{j} q^j = r(r-1)^t r^{-k} p^t r^{-t(k-1)} \prod_{i=1}^{t} \sum_{j=0}^{t+\omega-3} \binom{k-1}{j} q^j \leq$$

$$\leq r^{-(t+1)(k-1)} q^t \prod_{i=1}^{t} \sum_{j=0}^{t+\omega-3} k^j q^j \leq r^{-(t+1)(k-1)} q^t (kq)^{t(t+\omega-2)}. \tag{47}$$

3.6 Third part of $\mathcal{F}$: the event $\mathcal{C}_2(e,u)$

We shall show that if the event $\mathcal{C}_2(e,u)$ happens then the sum $\sum_{s\in e} I\{\xi_s \neq u\}$ cannot be very small. We shall establish the equality

$$\mathcal{C}_2(e,u) = \mathcal{C}_2(e,u) \cap \left\{\sum_{s\in e} I\{\xi_s \neq u\} \geq t+\omega-1\right\}. \tag{48}$$

Indeed, let us consider the intersection of three events (see the definition of the event $\mathcal{C}_2(e,u)$ in (36)):

$$\mathcal{C}_2(e,u) \cap \left\{ \sum_{s\in e} I\{\xi_s \neq u\} \leq t+\omega-2 \right\} =$$

$$= \bigcap_{s\in e} \{\zeta_s = u\} \cap \bigcap_{a=1}^{r} \overline{\mathcal{M}(e,a)} \cap \left\{ \sum_{s\in e} I\{\xi_s \neq u\} \leq t+\omega-2 \right\}.$$

The second and the third events imply the happening of the event $\mathcal{AM}(e,u)$ (see (34)). The first one implies that for every $s \in e$ satisfying $\xi_s \neq u$, we have $\zeta_s = \eta_s = u$. Moreover, since the event $\mathcal{AM}(e,u)$ holds, the set of such vertices in not empty. Consider a vertex $v \in e$ satisfying $\xi_v \neq u$ and $\xi_s = u$ for every $s \in e,\ s > v$. It is clear that the event $\mathcal{A}_v$ holds. So, $\zeta_v = \xi_v \neq u$, and we get a contradiction with the first event in the intersection. Thus, these three events are inconsistent, and we prove the equality (48).

Let us estimate the probability of $\mathcal{C}_2(e,u)$. Consider the random set $T = \{s \in e:\ \xi_s \neq u\}$. The event $\mathcal{C}_2(e,u)$ implies, first, that all $v \in T$ satisfy $\zeta_v = \eta_v = u$, and second, that $|T| \geq t+\omega-1$ (see (48)). Let us use the denotation: $E(e) = \{f \in E\backslash\{e\}:\ f \cap e \neq \emptyset\}$.

If $\zeta_v \neq \xi_v$ for some vertex v, then there should happen at least two events: the event D_v and the event

$$\mathcal{B}(e, f_v, v, u, a_v) = \left\{ \mathcal{M}(f_v, a_v) \cap \bigcap_{s\in f_v:\, s<v} \{\zeta_s = a_v\} \cap \{\zeta_v = \eta_v = u\} \right\}, \tag{49}$$

where f_v is some edge, satisfying $v \in e \cap f_v$, v is the first vertex from $e \cap f_v$ and $a_v \neq u$ is some color. It is clear that edges f_v are different for different v.

Let Y be an arbitrary subset of the edge e satisfying $y = |Y| \geq t+\omega-1$. Then we have the inclusion

$$\mathcal{C}_2(e,u) \cap \{T = Y\} \subset \bigcap_{s\in e\backslash Y} \{\xi_s = u\} \cap \bigcup_{\substack{a_1,\ldots,a_y=1 \\ a_i\neq u}}^{r} \bigcup_{\substack{f_1,\ldots,f_y\in E(e) \\ v_i\, \in\, f_i\cap e}} \bigcap_{i=1}^{y} \mathcal{B}(e, f_i, v_i, u, a_i), \tag{50}$$

where v_i is the first vertex from $f_i \cap e$. Since $y \geq t+\omega-1$ and the edge e is contained in at most ω 3-cycles, there is a set of edges $\{f'_1, \ldots, f'_{t-1}\} \subset \{f_1, \ldots, f_y\}$ such that the sets $\widehat{f'_i} = f'_i \setminus e$, $i = 1\ldots, t-1$, are pairwise disjoint and, moreover, the first vertices of $f_i \cap e$ are different for different $i = 1, \ldots, t-1$.

For further convenience, we introduce a notation of *the configuration of the second type*. For given edge e, an unordered set of edges $F = \{f_1, \ldots, f_{t-1}\}$ is said to be the configuration of the second type (denotation: $F \in \mathrm{CONF2}(e)$) if, for any any $i = 1, \ldots, t$, $f_i \in E(e)$ and, moreover, all the sets $f_i \setminus e$ are pairwise disjoint. Let us also use the following denotation:

$$S(F) = \bigcup_{i=1}^{t-1} (f_i \cap e)\,,$$

where $F = \{f_1, \ldots, f_{t-1}\} \in \mathrm{CONF2}(e)$.

Due to (50) and the above argument, we have the following inclusion:

$$\mathcal{C}_2(e,u) \subset \bigcup_{\substack{F\in\,\mathrm{CONF2}(e)\\ F=\{f_1,\ldots,f_{t-1}\}}} \left\{ \bigcap_{s\in\, e\setminus S(F)} \{\{\xi_s = u\} \sqcup \{\xi_s \neq u,\ \eta_s = u\}\} \cap \right.$$

$$\left. \cap \bigcap_{s\in\, S(F)} \{\eta_s = u\} \cap \bigcup_{\substack{a_1,\ldots,\, a_{t-1}=1\\ a_i\neq u}}^{r} \bigcap_{i=1}^{t-1} \mathcal{B}(e, f_i, v_i, u, a_i) \right\}, \tag{51}$$

where the set of edges $\{f_1, \ldots, f_t\}$ is assumed to be ordered according to the originally selected ordering of E, and, moreover, for any $i = 1, \ldots, t-1$, v_i denotes the first vertex in $f_i \cap e$.

The event $\mathcal{B}(e, f_i, v_i, u, a_i)$ is obviously (see (49)) contained in the event $\bigcap_{s\in f_i}\{\xi_s = a_i\} \cap \{\eta_{v_i} = u\}$. Hence, by (51) we get the relation

$$\mathcal{C}_2(e,u) \subset \bigcup_{\substack{F\in\,\mathrm{CONF2}(e)\\ F=\{f_1,\ldots,f_{t-1}\}}} \left\{ \bigcap_{s\in\, e\setminus S(F)} \{\{\xi_s = u\} \sqcup \{\xi_s \neq u,\ \eta_s = u\}\} \cap \right.$$

$$\left. \cap \bigcap_{s\in\, S(F)} \{\xi_s \neq u,\ \eta_s = u\} \cap \bigcup_{\substack{a_1,\ldots,\, a_{t-1}=1\\ a_i\neq u}}^{r} \bigcap_{i=1}^{t-1} \bigcap_{s\in f_i} \{\xi_s = a_i\} \right\}.$$

Let us take the union of both parts over u:

$$\bigcup_{u=1}^{r} \mathcal{C}_2(e,u) \subset \bigcup_{\substack{F\in\,\mathrm{CONF2}(e)\\ F=\{f_1,\ldots,f_{t-1}\}}} \bigcup_{u=1}^{r} \left\{ \bigcap_{s\in\, e\setminus S(F)} \{\{\xi_s = u\} \sqcup \{\xi_s \neq u,\ \eta_s = u\}\} \cap \right.$$

$$\left. \cap \bigcap_{s\in\, S(F)} \{\eta_s = u\} \cap \bigcup_{\substack{a_1,\ldots,\, a_{t-1}=1\\ a_i\neq u}}^{r} \bigcap_{i=1}^{t-1} \bigcap_{s\in f_i} \{\xi_s = a_i\} \right\}. \tag{52}$$

Let us introduce the following event:

$$\mathcal{Q}_3(e,F) = \bigcup_{u=1}^{r} \left\{ \bigcap_{s\in\, e\setminus S(F)} \{\{\xi_s = u\} \sqcup \{\xi_s \neq u,\ \eta_s = u\}\} \cap \right.$$

$$\left. \cap \bigcap_{s\in\, S(F)} \{\eta_s = u\} \cap \bigcup_{\substack{a_1,\ldots,\, a_{t-1}=1\\ a_i\neq u}}^{r} \bigcap_{i=1}^{t-1} \bigcap_{s\in f_i} \{\xi_s = a_i\} \right\}, \tag{53}$$

where e is an edge of H, $F = \{f_1, \dots, f_{t-1}\} \in \mathrm{CONF2}(e)$, and the edges are written according to the original ordering. By (52) and (53) we get the relation

$$\bigcup_{u=1}^{r} \mathcal{C}_2(e,u) \subset \bigcup_{F \in \ \mathrm{CONF2}(e)} \mathcal{Q}_3(e,F). \tag{54}$$

Now we are going to estimate the probability of $\mathcal{Q}_3(e,F)$.

Let us consider more closely the set of edges $F = \{f_1, \dots, f_{t-1}\}$. The hypergraph $H(F) = (V, F)$ can be divided into some number of connected components. Suppose $H_1, \dots, H_l$ are these components. Since F is a configuration of the second type, the edges f_i and f_j can have common vertices only inside the edge e. Moreover, we know that H is 2-simple. For every component H_j, $j = 1, \dots, l$, let us use the denotations:

$$h_j = |\{f \in E(H_j): \ |f \cap e| = 2\}| \text{ and } l_j = |\{f \in H_j: \ |f \cap e| = 1\}|.$$

Due to the 2-simplicity of H we have

$$\sum_{j=1}^{l} (h_j + l_j) = t - 1. \tag{55}$$

For any component H_j, let G_j be the following graph: $G_j = (V_j, E_j)$, where

$$V_j = e \cap V(H_j), \quad E_j = \{f \cap e: \ f \in E(H_j) \text{ and } |f \cap e| = 2\}|.$$

The following claim clarifies the structure of the configurations of the second type.

Claim 1. *For any $j = 1, \dots, l$, G_j is either a tree or an isolated vertex.*

Proof. If $h_j = 0$ then $l_j > 0$. But in this case H_j consists of only one edge f. Indeed, if $g \in E(H_j)$, $g \neq f$, then $|g \cap e| = 1$ and, moreover, $|g \cap f| > 0$. This can only happens when $f \cap e = g \cap e$, i.e. g has the same first vertex in its intersection with e as f. This fact is in conflict with the definition of the configuration of the second type. So, G_j is just an isolated vertex.

Now let $h_j > 0$. Since H_j is connected and $F \in \mathrm{CONF2}(e)$, G_j is also connected. Suppose there is a cycle $(w_1, \dots, w_m)$, $m \geq 3$, in G_j, i.e. $\{w_i, w_{i+1}\} \in E(G_j)$, $i = 1, \dots, m-1$ and, moreover, $\{w_1, w_m\} \in E(G_j)$. Without loss of generality, assume that $w_1 < w_j$ for any $j > 1$, i.e. w_1 is the vertex with the least number in the cycle. Since $\{w_1, w_2\} \in E(G_j)$, there is an edge $g_1 \in E(H_j)$ such that $\{w_1, w_2\} = g_1 \cap e$, so w_1 is the first vertex in $g_1 \cap e$. By analogy, there is another edge $g_2 \in E(H_j)$ such that $\{w_1, w_m\} = g_2 \cap e$, so w_1 is also the first vertex in $g_2 \cap e$. We obtain a contradiction with the fact that $F \in \mathrm{CONF2}(e)$. Hence, G_j is a tree. $\square$

Claim 1 implies that $|V(G_j)| = |E(G_j)| + 1 = h_j + 1$, thus,

$$|S(F)| = \sum_{j=1}^{l} |V(G_j)| = \sum_{j=1}^{l} (h_j + 1). \tag{56}$$

Moreover, from the definitions of the values h_j and l_j we get that, for any $j = 1, \dots, l$,

$$\left| \bigcup_{f \in H_j} f \right| = (k-2)h_j + (k-1)l_j + |V(G_j)| = (k-1)(h_j + l_j) + 1. \tag{57}$$

Finally, Claim 1 implies that, for any $j = 1, \dots, l$,

$$l_j \le 1. \tag{58}$$

Indeed, since there is a bijection between the edges of H_j and the first vertices in their intersection with e, we have

$$h_j + l_j = |E(H_j)| \le |V(G_j)| = h_j + 1,$$

and the inequality (58) follows.

Using the notations introduced above one can easily find the probability of the event $\mathcal{Q}_3(e, F)$ (see (53)):

$$\mathsf{P}\left(\mathcal{Q}_3(e,F)\right) = r\left(\frac{1}{r} + \frac{q}{r}\right)^{k-|S(F)|} p^{|S(F)|}(r-1)^l \prod_{j=1}^{l} r^{-\left|\bigcup_{f\in H_j} f\right|}. \tag{59}$$

Let us explain the last two factors in the right-hand side of (59). Since all the edges of F are monochromatic in the main coloring ξ, the values of ξ_s should coincide for all $s \in V(H_j)$. Thus, we only have to choose a color (not equal to u) for every component (the factor $(r-1)^l$). The last factor is equal to the probability (we have already chosen the colors) that every edge in the component H_j is monochromatic in the main coloring ξ.

Using obtained relations (55), (56), (57), (58), we get the following estimate of the probability of the event $\mathcal{Q}_3(e, F)$:

$$\mathsf{P}\left(\mathcal{Q}_3(e,F)\right) = r\left(\frac{1}{r} + \frac{q}{r}\right)^{k-|S(F)|} p^{|S(F)|}(r-1)^l \prod_{j=1}^{l} r^{-\left|\bigcup_{f\in H_j} f\right|} =$$

$$= r^{1-k}(1+q)^{k-|S(F)|}(rp)^{|S(F)|}(r-1)^l \prod_{j=1}^{l} r^{-(k-1)(h_j+l_j)-1} =$$

$$\begin{aligned} &= r^{1-k}(1+q)^{k-|S(F)|}(rp)^{|S(F)|}(r-1)^l r^{-(k-1)(t-1)-l} \le \\ &\le r^{(1-k)t}(1+q)^k (rp)^{|S(F)|} \le r^{(1-k)t}(1+q)^k(2q)^{t-1}. \end{aligned} \tag{60}$$

We need to comment only the last inequality. From the condition (32) we have $2q \le 1$ and, moreover, we know that $rp = (r/(r-1))q \le 2q$. Finally, from (55), (57) and (58) we immediately see that $|S(F)| \ge t - 1$.

The bound (60) completes the estimation of different parts of the event $\mathcal{F}$. Now we shall prove that the probability of $\mathcal{F}$ is less than 1 under the conditions of Theorem 5.

3.7 Application of Local Lemma for estimating the probability of $\mathcal{F}$

Remember that by the definitions (35) and (36) of the events $\mathcal{F}$ and $\mathcal{C}_i(e, u)$, $i = 1, 2, 3$, $e \in E$, $u = 1, \dots, r$, we have the equality

$$\mathcal{F} = \bigcup_{e\in E} \bigcup_{u=1}^{r} \left(\mathcal{C}_1(e,u) \cup \mathcal{C}_2(e,u) \cup \mathcal{C}_3(e,u)\right).$$

It follows from the obtained relations (38), (40), (41), (46) and (54), that

$$\mathcal{F} \subset \bigcup_{e \in E} \{\mathcal{Q}_0(e) \cup \mathcal{Q}_1(e)\} \cup \bigcup_{e \in E} \bigcup_{F \in \mathrm{CONF1}(e)} \mathcal{Q}_2(e,F) \cup \bigcup_{e \in E} \bigcup_{F \in \mathrm{CONF2}(e)} \mathcal{Q}_3(e,F). \tag{61}$$

Further, we shall use a classical claim, which is called Local Lemma. This statement was first proved in the paper of P. Erdős and L. Lovász [8]. We shall formulate it in a special case.

Theorem 6. *Let events $\mathcal{B}_1, \ldots, \mathcal{B}_N$ be given on some probability space. Let $S_1, \ldots, S_N$ be subsets of $\mathcal{R}_N = \{1, \ldots, N\}$ such that for any $i = 1, \ldots, N$, the event $\mathcal{B}_i$ is independent of the algebra generated by the events $\{\mathcal{B}_j, j \in \mathcal{R}_N \backslash S_i\}$. If, for any $i = 1, \ldots, N$, the following inequality holds*

$$\sum_{j \in S_i} \mathsf{P}(\mathcal{B}_j) \le 1/4, \tag{62}$$

then $\mathsf{P}\left(\bigcap_{j=1}^{N} \overline{\mathcal{B}_j}\right) \ge \prod_{j=1}^{N} (1 - 2\mathsf{P}(\mathcal{B}_i)) > 0.$

The proof of the Local Lemma can be found in the book [19]. Consider the system of events Ψ consisting of all the events $\mathcal{Q}_i(e)$, $i = 0, 1$, $e \in E$, the events $\mathcal{Q}_2(e,F)$, $e \in E$, $F \in \mathrm{CONF1}(e)$, and also the events $\mathcal{Q}_3(e,F)$, $e \in E$, $F \in \mathrm{CONF2}(e)$. By (61) the inequality holds

$$\mathsf{P}(\mathcal{F}) \le \mathsf{P}\left(\bigcup_{\mathcal{B} \in \Psi} \mathcal{B}\right) = 1 - \mathsf{P}\left(\bigcap_{\mathcal{B} \in \Psi} \overline{\mathcal{B}}\right). \tag{63}$$

We shall show that the probability of $\bigcap_{\mathcal{B} \in \Psi} \overline{\mathcal{B}}$ is greater than zero. Due to Local Lemma (see Theorem 6), it is sufficient to find, for every $\mathcal{B} \in \Psi$, a system of events $\Psi_{\mathcal{B}} \subset \Psi$ such that $\mathcal{B}$ and the algebra generated by $\{\mathcal{Q} \in \Psi \backslash \Psi_{\mathcal{B}}\}$ are independent, and, moreover, such that the following inequality holds:

$$\sum_{\mathcal{Q} \in \Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{Q}) \le 1/4. \tag{64}$$

The event $\mathcal{B} \in \Psi$ can have one of the following three types:

1. $\mathcal{B} = \mathcal{Q}_i(e)$ for some $e \in E$ and $i \in \{0, 1\}$;
2. $\mathcal{B} = \mathcal{Q}_2(e,F)$ for some $e \in E$ and $F \in \mathrm{CONF1}(e)$;
3. $\mathcal{B} = \mathcal{Q}_3(e,F)$ for some $e \in E$ and $F \in \mathrm{CONF2}(e)$.

For any $\mathcal{B} \in \Psi$, we define *the domain* $D(\mathcal{B})$ of the event $\mathcal{B}$ as follows:

$$D(\mathcal{B}) = \begin{cases} e, & \text{if } \mathcal{B} = \mathcal{Q}_i(e),\ i = 0, 1; \\ e \cup \left(\bigcup_{f \in F} f\right), & \text{if } \mathcal{B} = \mathcal{Q}_i(e,F),\ i = 2, 3. \end{cases}$$

By the definitions (38), (41), (45), (53) the event $\mathcal{B}$ belongs to the algebra generated by the random variables $\{\xi_j, \eta_j : \ j \in D(\mathcal{B})\}$. Then this event is independent of the algebra generated by the random variables $\{\xi_j, \eta_j : \ j \in V \setminus D(\mathcal{B})\}$. We take the system $\Psi_{\mathcal{B}}$ consisting of all the events $\mathcal{Q} \in \Psi$ such that the domains of $\mathcal{Q}$ and $\mathcal{B}$ have nonempty intersection. In other words,

$$\Psi_{\mathcal{B}} = \{\mathcal{Q} \in \Psi : \quad D(\mathcal{Q}) \cap D(\mathcal{B}) \neq \emptyset\}.$$

Thus, the event $\mathcal{B}$ is independent of the algebra generated by $\{\mathcal{J} \in \Psi \backslash \Psi_{\mathcal{B}}\}$, if we choose $\Psi_{\mathcal{B}}$ in this way. We have to check the inequality (64). By the choice of the set $\Psi_{\mathcal{B}}$ we get the relation

$$\sum_{\mathcal{J} \in \Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{J}) \leq \sum_{e \in E:\, e \cap D(\mathcal{B}) \neq \emptyset} (\mathsf{P}(\mathcal{Q}_0(e)) + \mathsf{P}(\mathcal{Q}_1(e))) + \sum_{\substack{e \in E,\ F \in \mathrm{CONF1}(e): \\ D(\mathcal{B}) \cap D(\mathcal{Q}_2(e,F)) \neq \emptyset}} \mathsf{P}(\mathcal{Q}_2(e,F)) +$$

$$+ \sum_{\substack{e \in E,\ F \in \mathrm{CONF2}(e): \\ D(\mathcal{B}) \cap D(\mathcal{Q}_3(e,F)) \neq \emptyset}} \mathsf{P}(\mathcal{Q}_3(e,F)). \tag{65}$$

Let us denote by $a(\mathcal{B})$, $b(\mathcal{B})$ and $c(\mathcal{B})$ the number of summands in the first sum, the second sum and the third sum in the right-hand side of (65) respectively. Using these denotations from the relation (65) and the estimates (39), (42), (47), (60) we get the inequality

$$\sum_{\mathcal{J} \in \Psi_{\mathcal{B}}} \mathsf{P}(\mathcal{B}) \leq a(\mathcal{B}) \left(r(r-1) \left(\frac{p}{r}\right)^k + r^{1-k}(1-q)^{k-t-\omega}(kq)^{t+\omega} \right) +$$

$$+ b(\mathcal{B}) r^{-(t+1)(k-1)} q^t (kq)^{t(t+\omega-2)} + c(\mathcal{B}) r^{-t(k-1)} (1+q)^k (2q)^{t-1}. \tag{66}$$

Now we shall consider three cases depending on $\mathcal{B}$.

1. $\mathcal{B} = \mathcal{Q}_i(e)$ for some $e \in E$ and $i \in \{0,1\}$. By the condition (30) of Theorem 5 there exist at most d other edges intersecting an arbitrary $e \in E$. So,

$$a(\mathcal{B}) \leq d+1, \ b(\mathcal{B}) \leq (d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1},$$

$$c(\mathcal{B}) \leq (d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}. \tag{67}$$

The first inequality in (67) is obvious. To show the last two it is sufficient to notice that e can intersect either with e' from the event $\mathcal{Q}_2(e', F)$ or with some $f \in F$.

2. $\mathcal{B} = \mathcal{Q}_2(e,F)$ for some $e \in E$ and $F \in \mathrm{CONF1}(e)$. This event depends on $(t+1)$ edges. So, by using the estimates from (67) we get

$$a(\mathcal{B}) \leq (t+1)(d+1), \ b(\mathcal{B}) \leq (t+1)\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right),$$

$$c(\mathcal{B}) \leq (t+1)\left((d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}\right). \tag{68}$$

3. $\mathcal{B} = \mathcal{Q}_3(e, F)$ for some $e \in E$ and $F \in$ CONF2(e). This event depends on t edges. So, as in the previous case

$$a(\mathcal{B}) \leq t(d+1),\ b(\mathcal{B}) \leq t\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right),$$

$$c(\mathcal{B}) \leq t\left((d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}\right). \tag{69}$$

It is easy to see from (67), (68) and (69) that the maximal upper bounds for $a(\mathcal{B})$, $b(\mathcal{B})$ and $c(\mathcal{B})$ are in the second case. So, to prove (64) it is sufficient to establish (due to (66)) the following inequality:

$$W = (t+1)(d+1)\left(r(r-1)\left(\frac{p}{r}\right)^k + r^{1-k}(1-q)^{k-t-\omega}(kq)^{t+\omega}\right) +$$
$$+(t+1)\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right) r^{-(t+1)(k-1)}q^t(kq)^{t(t+\omega-2)} +$$
$$+ (t+1)\left((d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}\right) r^{-t(k-1)}(1+q)^k(2q)^{t-1} \leq 1/4 \tag{70}$$

We shall need some additional estimates contained in the next section.

3.8 Auxiliary analytics

The value W (see (70)) consists of four summands:

$$(t+1)(d+1)r(r-1)\left(p/r\right)^k, \quad (t+1)(d+1)r^{1-k}(1-q)^{k-t-\omega}(kq)^{t+\omega},$$
$$(t+1)\left((d+1)\binom{d}{t} + (d+1)d\binom{d-1}{t-1}\right) q^t r^{-(k-1)(t+1)}(kq)^{t(t+\omega-2)},$$
$$(t+1)\left((d+1)\binom{d}{t-1} + (d+1)d\binom{d-1}{t-2}\right) r^{-(k-1)t}(1+q)^k(2q)^{t-1}.$$

Consider and estimate them separately.

1. The first summand is $(t+1)(d+1)r(r-1)\left(p/r\right)^k$. Using the restriction (30), the conditions (31) and (32) we obtain the upper bound for the first summand:

$$(t+1)(d+1)r(r-1)\left(\frac{p}{r}\right)^n \leq (t+1)kr^{k-1}r^{1-n}(r-1)\left(\frac{q}{r-1}\right)^k =$$
$$= (t+1)k(r-1)^{1-k}q^k \leq k^2q^k \leq k^2 2^{-k}. \tag{71}$$

2. The second summand is $(t+1)(d+1)r^{1-k}(1-q)^{k-t-\omega}(kq)^{t+\omega}$. Since the choice of parameter q in (29), we get the relations

$$(t+1)(d+1)r^{1-k}(1-q)^{k-t-\omega}(kq)^{t+\omega} \leq (t+1)kr^{k-1}r^{1-k}(1-q)^{k-t-\omega}(kq)^{t+\omega} =$$
$$= (t+1)n(1-q)^{k-t-\omega}\left(\alpha \ln k\right)^{t+\omega} \leq (t+1)ke^{q(t+\omega)-qk}\left(\alpha \ln k\right)^{t+\omega} =$$
$$= (t+1)k^{1-\alpha}e^{\alpha(\ln k)(t+\omega)/k}\left(\alpha \ln k\right)^{t+\omega}. \tag{72}$$

3. Let us consider the third summand in the expression (70) for the value W:

$$(t+1)\left((d+1)\binom{d}{t}+(d+1)d\binom{d-1}{t-1}\right)q^t r^{-(k-1)(t+1)}(kq)^{t(t+\omega-2)}. \tag{73}$$

We shall need some preliminary estimates.

First, the following inequalities hold:

$$(t+1)\left((d+1)\binom{d}{t}+(d+1)d\binom{d-1}{t-1}\right)=(t+1)\binom{d}{t}(d+1)(t+1)\leq$$
$$\leq (t+1)^2(d+1)\frac{d^t}{t!}\leq (t+1)^2\frac{(d+1)^t}{t!}. \tag{74}$$

Second, the choice of parameters t and q (see (29)) implies the relations

$$q^t(kq)^{t(t+\omega-2)}=k^{-t}(kq)^{t(t+\omega-1)}\leq k^{-t}(kq)^{t^2+t\omega}=k^{-t}\exp\left\{t^2\ln(\alpha\ln k)\right\}(kq)^{t\omega}\leq$$
$$\leq k^{-t}\exp\{\ln k\}(\alpha\ln k)^{t\omega}=k^{1-t}(\alpha\ln k)^{t\omega}. \tag{75}$$

Finally, from (74), (75) and the original restriction (30), we obtain the upper bound for the expression (73):

$$(t+1)\left((d+1)\binom{d}{t}+(d+1)d\binom{d-1}{t-1}\right)q^t r^{-(k-1)(t+1)}(kq)^{t(t+\omega-2)}\leq$$
$$\leq\frac{(t+1)^2}{t!}(d+1)^{t+1}r^{-(k-1)(t+1)}k^{1-t}(\alpha\ln k)^{t\omega}\leq\frac{(t+1)^2}{t!}k^{(t+1)(1-b/t)}k^{1-t}(\alpha\ln k)^{t\omega}\leq$$
$$\leq\frac{(t+1)^2}{t!}k^{t+1-(b(t+1)/t)}k^{1-t}(\alpha\ln k)^{t\omega}=$$
$$=\frac{(t+1)^2}{t!}k^{2-b-(b/t)}(\alpha\ln k)^{t\omega}\leq\frac{(t+1)^2}{t!}k^{2-b}(\alpha\ln k)^{t\omega}. \tag{76}$$

4. It remains to estimate the fourth summand in the expression (70) for the value W:

$$(t+1)\left((d+1)\binom{d}{t-1}+(d+1)d\binom{d-1}{t-2}\right)r^{-(k-1)t}(1+q)^k(2q)^{t-1}. \tag{77}$$

By an analogy with (74), we get:

$$(t+1)\left((d+1)\binom{d}{t-1}+(d+1)d\binom{d-1}{t-2}\right)=(t+1)\binom{d}{t-1}(d+1)t\leq$$
$$\leq(t+1)t\left(\frac{de}{t-1}\right)^{t-1}(d+1)\leq(d+1)^t\left(\frac{e}{t-1}\right)^{t-1}(t+1)t. \tag{78}$$

Further, by (29) it holds that

$$(2q)^{t-1}(1+q)^k\leq k^{1-t}(2\alpha\ln k)^{t-1}e^{qk}=k^{1+\alpha-t}(2\alpha\ln k)^{t-1}. \tag{79}$$

Finally, from (78), (79) and (30) we obtain an upper bound for the expression (77):

$$(t+1)\left((d+1)\binom{d}{t-1}+(d+1)d\binom{d-1}{t-2}\right)r^{-(k-1)t}(1+q)^k(2q)^{t-1}\leq$$

$$\leq (t+1)t\left(\frac{e}{t-1}\right)^{t-1}(d+1)^t r^{-(k-1)t}k^{1+\alpha-t}(2\alpha\ln k)^{t-1}\leq$$

$$\leq (t+1)t\left(\frac{2e\alpha\ln k}{t-1}\right)^{t-1} r^{(k-1)t}\,k^{t(1-b/t)}r^{-(k-1)t}k^{1+\alpha-t}=$$

$$=(t+1)t\left(\frac{2e\alpha\ln k}{t-1}\right)^{t-1}k^{1+\alpha-b}. \tag{80}$$

The inequality (80) completes the estimation of the parts of the value W.

3.9 The completion of the proof of Theorem 5

Let us gather the obtained bounds for the summands in the expression (70) for the value W. The relations (71), (72), (76) and (80) imply the inequalities

$$W\leq\frac{k^2}{2^k}+(t+1)k^{1-\alpha}e^{\alpha(\ln k)(t+\omega)/k}\left(\alpha\ln k\right)^{t+\omega}+\frac{(t+1)^2}{t!}k^{2-b}(\alpha\ln k)^{t\omega}+$$

$$+(t+1)t\left(\frac{2e\alpha\ln k}{t-1}\right)^{t-1}k^{1+\alpha-b}<\frac{1}{4},$$

the last of which holds, since the condition (33) of theorem 5. Thus, the required relation (70) is established. It implies the inequality (64) necessary for the application of Local Lemma. It follows from Local Lemma that the probability of simultaneous happening of all the events $\overline{\mathcal{B}}$, where $\mathcal{B}\in\Psi$, is greater than zero. Then by (63) we have shown that

$$\mathsf{P}\left(\mathcal{F}\right)<1.$$

Let us complete the proof. Indeed, we have proved, that the probability of the event that the random coloring $\vec{\zeta}$ is not a proper coloring of H is less than one. So, $\vec{\zeta}$ is a proper coloring with positive probability, and $\chi(H)\leq r$. Theorem 5 is proved.

3.10 The completion of the proof of Theorem 4

We shall use Theorem 5. Let us choose the parameters b and α:

$$b=4,\quad \alpha=2.$$

By this choice of b, α and the condition $\omega\leq\sqrt{\ln k/(\ln\ln k)}$ there exists an integer k_1 such that for all $k\geq k_1$, the inequalities (31) and (32) hold. Let us consider the left part of (33). We have $t=O\left(\sqrt{\ln k/\ln\ln k}\right)$ (see (29)), so

$$(t+1)k^{1-\alpha}e^{\alpha(\ln k)(t+\omega)/k}\left(\alpha\ln k\right)^{t+\omega}=e^{O(\ln\ln k)}k^{-1}e^{o(1)}e^{O\left(\sqrt{\ln k\ln\ln k}\right)}=o(1),\ k\to\infty,$$

$$\frac{(t+1)^2}{t!}\,k^{2-b}(\alpha\ln k)^{t\omega} = O\left(k^{-2}\right)e^{\ln k(1+o(1))} = o(1),\ k\to\infty,$$
$$(t+1)t\left(\frac{2e\alpha\ln k}{t-1}\right)^{t-1}k^{1+\alpha-b} = e^{O\left(\sqrt{\ln k\ln\ln k}\right)}k^{-1} = o(1),\ k\to\infty.$$

These relations imply the existence of an integer k_2 such that the inequality (33) holds, for all $k \geq k_2$.

Let $H = (V,E)$ be an k-uniform hypergraph, $H \in \mathcal{H}(k,r,\omega)$ with $\omega \leq \sqrt{\ln k/(\ln\ln k)}$. In the case $k \geq k_0 = \max(k_1,k_2)$ the hypergraph H satisfies all the conditions of Theorem 5, except (30). But H is not r-colorable, and so there exists an edge $e \in E$ with edge degree at least $\lfloor r^{k-1}k^{1-b/t}\rfloor$. So, the edge e contains a vertex with degree at least

$$\left\lfloor r^{k-1}k^{1-b/t}\right\rfloor/k + 1 \geq \left(r^{k-1}k^{1-b/t}-1\right)/k+1 = r^{k-1}k^{-b/t}+1-1/k.$$

Thus, we have established the inequality $\Delta(H) > r^{k-1}k^{-b/t}$ and, consequently,

$$\Delta(\mathcal{H}(n,r,\omega)) \geq r^{k-1}k^{-b/t} = r^{k-1}k^{-4\left\lfloor\sqrt{\frac{\ln k}{\ln(2\ln k)}}\right\rfloor^{-1}}.$$

Theorem 4 is proved.

4 Choosability in random hypergraphs

In this section we will discuss r-choosability of the random hypergraph $H(n,k,p)$. Let us recall the required definitions.

We say that a hypergraph H is *r-choosable* if for every family of sets $L = \{L(v) :\ v \in V\}$ (L is called *list assignment*), such that $|L(v)| = r$ for all $v \in V$, there is a proper coloring from the lists (for every $v \in V$ we should use a color from $L(v)$). The *choice number* of a hypergraph H, denoted by $ch(H)$, is the least r such that H is r-choosable. It is clear that $\chi(H) \leq ch(H)$. The choice numbers of graphs were independently introduced by V.G. Vizing (see [20]) and by P. Erdos, A. Rubin and H. Taylor (see [21]). In this paper we consider the threshold probability for r-choosability of $H(n,k,p)$.

4.1 Threshold for r-choosability in $H(n,k,p)$

The choice number of the random hypergraph $H(n,k,p)$ has been studied by M. Krivelevich and V. Vu (see [22]). They proved that $ch(H(n,k,p))$ is asymptotically very closed to $\chi(H(n,k,p))$. Their first result holds for almost all p, but has at little gap between $ch(H(n,k,p))$ and $\chi(H(n,k,p))$.

Theorem 7. (M. Krivelevich, V. Vu, [22]) *Suppose $k \geq 2$ is fixed. There exists a constant $C = C(k)$ such that, for any p satisfying $C\,n^{1-k} \leq p \leq 0.9$, the following convergence holds*

$$\mathsf{P}\left(ch(H(n,k,p)) \leq (1+\psi(n))\,k^{1/(k-1)}\,\chi(H(n,k,p))\right) \to 1 \ \textit{as}\ n\to\infty,$$

where $\psi(n) \to 0$ as $n^{k-1}p \to \infty$.

The second theorem from [22] states that, for sufficiently large p, this gap can be removed.

Theorem 8. (M. Krivelevich, V. Vu, [22]) *Suppose $k \geq 2$ is fixed and $0 < \varepsilon < (k-1)^2/(2k)$. Then, for any p satisfying $n^{-(k-1)^2/(2k)+\varepsilon} \leq p \leq 0.9$, the following convergence holds*

$$\mathsf{P}\left(ch(H(n,k,p)) = (1+o(1))\,\chi(H(n,k,p))\right) \to 1 \text{ as } n \to \infty.$$

Theorem 7 together with Theorem 2 of Krivelevich and Sudakov implies the following corollary, which is an analogue of Corollary 1 for r-choosability.

Corollary 3. *Let $k \geq 3$ and $\varepsilon \in (0,1)$ be fixed. There is a constant $r_0 = r_0(k,\varepsilon)$ such that, for any $r = r(n)$ satisfying the conditions*

$$r \geq r_0, \quad r^{k-1} \ln r = o(n^{k-1}),$$

the following convergence holds:

$$\mathsf{P}\left(ch(H(n,k,p)) \leq r\right) \to 1, \text{ where } p = (1-\varepsilon)\,\frac{r^{k-1}\ln r}{k}\,\frac{n}{\binom{n}{k}}.$$

We see that the lower bound for the threshold for r-choosability provided by Corollary 3 does not coincide with the upper bound in Lemma 2 (if hypergraph is not r-colorable then it is also not r-choosable). Their ratio has an order of k. Recall that applying Theorem 2 provides the following restrictions on the parameters r and k in Corollary 3 (see (9)):

$$r = \Omega\left(k^{29}(\ln k)^{28}\right), \quad n \geq k^{9k+O(k\ln\ln k/(\ln k))}. \tag{81}$$

What can be said about the lower bound for the threshold for r-choosability when $r = O\left(k^{29}(\ln k)^{28}\right)$?

Remark 3. It should be noted that, for very large r (e.g., $r > \sqrt{n}$) and fixed k, Theorem 8 together with Theorem 2 gives an asymptotic value for the required threshold for r-choosability:

$$p^* \sim r^{k-1}\ln r\,\frac{n}{\binom{n}{k}}.$$

The proof of Theorem 1 by Achlioptas, Krivelevich, Kim and Tetali is based on the deterministic coloring algorithm, which cannot be generalized to the case of an arbitrary r-uniform list assignment, so the lower bound (7) does not hold for the threshold probability for r-choosability. Moreover, the proof of the result (4) by Achlioptas and Moore is also cannot be adopted for list colorings. Thus, in the case when r is small in comparison with k we have only the result of Lemma 1 which is just a generalization of the result (1) by Alon and Spencer.

Lemma 4. *There exists an integer k_0 and a positive number c such that for any fixed $k \geq k_0$ and $r \geq 2$, the following statement holds:*

$$\text{if } p \leq c\,\frac{r^{k-1}}{k^2}\,\frac{n}{\binom{n}{k}}, \text{ then } \mathsf{P}\left(H(n,k,p) \text{ is } r\text{-choosable}\right) \to 1. \tag{82}$$

Our new results concerning r-choosability in random hypergraphs are formulated in the following two theorems.

Theorem 9. *Suppose $k = k(n) \geq 3$ and $r = r(n) \geq 3$ satisfy the relation*

$$\frac{3}{128}\frac{r^{k-1}}{\sqrt{k}} - \ln n \to -\infty \text{ as } n \to \infty.$$

If

$$p \leq \frac{3}{32}\frac{r^{k-1}}{k^{3/2}}\frac{n}{\binom{n}{k}}, \tag{83}$$

then $\mathsf{P}\left(H(n,k,p) \text{ is } r\text{-choosable}\right) \to 1$.

Theorem 10. *Suppose $\delta \in (0,1)$ is a constant. Let $k = k(n)$ and $r = r(n) \geq 2$ satisfy the following conditions: $k \geq k_0$, where k_0 is some absolute constant, and, moreover,*

$$(k-1)\ln r < \frac{1-\delta}{2}\ln n, \quad r^{k-1}k^{-\varphi(k)} \geq 6\ln n,$$

where $\varphi(k) = 4\left\lfloor\sqrt{\frac{\ln k}{\ln(2\ln k)}}\right\rfloor^{-1}$. *Then for function $p = p(n)$, satisfying*

$$p \leq \frac{1}{2}\frac{r^{k-1}}{k^{1+\varphi(k)}}\frac{n}{\binom{n}{k}}, \tag{84}$$

we have $\mathsf{P}\left(H(n,k,p) \text{ is } r\text{-choosable}\right) \to 1$ *as* $n \to \infty$.

It is easy to see that Theorem 9 and Theorem 10 stated that the results of Corollary 2 (assertion 1)) and Theorem 3 also hold in the case of list colorings. Both provided bounds are better (for all sufficiently large k) than the result (82) of Lemma 4, and both are worse than the result of Corollary 3. Hence, Theorem 10 gives the best lower bound (18) for the threshold probability for r-choosability of $H(n,k,p)$ in the wide area of the parameters (recall the restriction (81)):

$$r \leq k^{29}(\ln k)^{28} \text{ and } 6\,k^{\varphi(k)}\ln n \leq r^{k-1} \leq n^{(1-\delta)/2},$$

where $k \geq k_0$ is sufficiently large. The inequality (17), in comparison with (18), does not have an upper restriction $r^{k-1} \leq n^{(1-\delta)/4}$, so it provides the best lower bound in the area

$$3 \leq r \leq k^{29}(\ln k)^{28} \text{ and } r^{k-1} \geq n^{(1-\delta)/2}.$$

The proofs of Theorems 9 and Theorem 10 are very similar to the proofs of the first assertion of Corollary 2 and Theorem 3, so we do not give the complete argument and describe only the main ides and differences.

4.2 Ideas of the proofs of Theorems 9 and 10

For given $k, r \geq 2$, let $\Delta_{list}(k,r)$ denote the minimum possible $\Delta(H)$, where H is a k-uniform non-r-choosable hypergraph. In [11] D.A. Shabanov shows that the lower bound (12) for $\Delta(k,r)$ (see §2.1) holds for $\Delta_{list}(k,r)$ also: for any $k, r \geq 3$,

$$\Delta_{list}(k,r) > \frac{1}{8}k^{-1/2}r^{k-1}.$$

Using this inequality one can easily prove Theorem 9 by the same argument as in Lemma 3.

Remark 4. The lower bound (13) for $\Delta(k,r)$ obtained by Kostochka, Kumbhat and Rödl does not hold for $\Delta_{list}(k,r)$, so we cannot apply it to r-choosability of random hypergraphs.

To prove Theorem 10 it is sufficient to show that under the conditions of Theorem 5 the hypergraph H is not only r-colorable, but is r-choosable. The proof of r-choosability remains almost the same. The difference appears in the distributions of the random variables.

Suppose $H=(V,E)$ is a k-uniform hypergraph satisfying the conditions of Theorem 5 and let $L=\{L(v):\ v\in V\}$ be an r-uniform list assignment with the set of colors $\mathbb{N}$. Without loss of generality, $V=\{1,\ldots,w\}$. In comparison with §3.3 we introduce random variables with another distribution. Let $\xi_1,\ldots,\xi_w$ and $\eta_1,\ldots,\eta_w$, be mutually independent random variables with the following distribution:

- ξ_i, $i=1,\ldots,w$, has the uniform distribution on the set $L(i)$ ($i=1,\ldots,w$),
- η_i, $i=1,\ldots,w$, takes all values from $L(i)$ with the same probability p and the value 0 with probability $1-rp$.

For given edge $e\in E$, let $M(e)$ be equal to $M(e)=\bigcap_{s\in e}L(s)$. For every $u\in M(e)$, we introduce the events $\mathcal{M}(e,u)$, $\mathcal{AM}(e,u)$ whose definitions are the same as in §3.3 (see (34)). Then we construct the random coloring $\vec{\zeta}=(\zeta_1,\ldots,\zeta_w)$ by the same way as in §3.3. The only difference is that in the definitions of the events $\mathcal{A}_i$ and $\mathcal{D}_i$ the parameter u does not take values from 1 to r, it should take values from an appropriate set $M(e)$ or $M(f)$. The rest of the proof remains the same without any unobvious change.

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